m denotes Lebesgue measure on $\mathbb{R}$. In some problems it denotes Lebesgue measure on $\mathbb{R}^n$.

Problem 1

Given $f \in C([0,\infty))$ such that $f(x) \to 0$ as $x \to \infty$ show that for any $\epsilon > 0$ there is a polynomial p such that $|f(x) - e^{-x}p(x)| < \epsilon \ \forall x \in [0,\infty)$.

[See also problem 109 below]

We give two proofs:

1. Let $g(x) = f(-\log(x)), 0 < x \leq 1$ and $g(0) = 0$. By Weierstrauss Approximation Theorem we can find a polynomial q such that $|g(x) - q(x)| < \epsilon/2$ for $0 \leq x \leq 1$. Note that the constant term c_0 in q satisfies the inequality $|c_0| < \epsilon/2$. If $q_1 = q - c_0$ then $|g(x) - q_1(x)| < \epsilon$ for $0 \leq x \leq 1$. We can write $q_1(x)$ as $\sum\limits_{j=1}^{N} c_j x^j$. We now have the inequality $\left| f(x) - \sum\limits_{j=1}^{N} c_j e^{-jx} \right| < \epsilon$ for all $x \geq 0$. From this we conclude that if the result holds for the functions $f(x) = e^{-jx}, j \in \mathbb{N}$ then it holds for the given function f. We now prove the result for these functions using induction on j.

For $j = 1$ we take $p = 1$. Suppose $\left| e^{-jx} - e^{-x}p(x) \right| < \epsilon \ \forall x \in [0,\infty)$ for some polynomial p. Then $\left| e^{-(j+1)x} - e^{-(1+\frac{1}{j})x} p(\frac{1+j}{j}x) \right| < \epsilon \ \forall x \in [0,\infty)$. it suffices to show that $\left| e^{-x}\phi(x)p(\frac{1+j}{j}x) - e^{-(1+\frac{1}{j})x} p(\frac{1+j}{j}x) \right| < \epsilon \ \forall x \in [0,\infty)$ for some polynomial ϕ. Let $\phi(x) = \sum\limits_{k=0}^{2N} \frac{(-x/j)^k}{k!}$ where N is a positive integer to be specified. We first note that $M \equiv \sup\{\left| e^{-(1-\frac{1}{j})x} p(\frac{1+j}{j}x) \right| : x \geq 0\} < \infty$ if $j > 1$. This is also true for $j = 1$ because $p(x) \equiv 1$ in this case.

Thus $\left| e^{-x}\phi(x)p(\frac{1+j}{j}x) - e^{-(1+\frac{1}{j})x} p(\frac{1+j}{j}x) \right| \leq M e^{-x/j} \left| \frac{(-x/j)^{2N}}{(2N)!} \right|$. Here we use that fact that $\sum\limits_{k=0}^{2N-1} \frac{(-x/j)^k}{k!} \leq e^{-x/j} \leq \sum\limits_{k=0}^{2N} \frac{(-x/j)^k}{k!}$ and hence $\left| e^{-x/j} - \phi(x) \right| \leq \left| \frac{(-x/j)^{2N}}{(2N)!} \right|$. The last expression attains its maximum at the point $x = 2jN$ and the maximum value is $Me^{-2N}\frac{(2N)^{2N}}{(2N)!}$. By Stirling's Formula [$\lim \frac{(2N)!}{e^{-2N}(2N)^{2N+1/2}} = \sqrt{2\pi}$] we see that desired inequality holds if N is sufficiently large.

Manjunath Krishnapur's solution:

Consider the Banach space of all continuous functions on $[0,\infty)$ vanishing at ∞ with the supremum norm. We have to show that the subspace $\{e^{-x}p(x) : p$ is a polynomial$\}$ is dense in this space. If not, then there is a continuous linear functional which vanishes on this subspace but not everywhere. By Riesz Representation Theorem there is a real measure μ such that $\int e^{-x}p(x)d\mu(x) = 0$

for every p but $\mu \neq 0$. Writing μ as $\mu_1 - \mu_2$ where μ_1 and μ_2 are positive finite measures we have $\int e^{-x}p(x)d\mu_1(x) = \int e^{-x}p(x)d\mu_2(x)$. Let $d\nu_1 = e^{-x}d\mu_1$ and $d\nu_2 = e^{-x}d\mu_2$. Then $\int p(x)d\nu_1(x) = \int p(x)d\nu_2(x)$ for every polynomial p but $\nu_1 \neq \nu_2$. Let $\phi_j(z) = \int e^{zt}d\nu_j(t), j = 1, 2$. These functions are holomorphic in $\{\operatorname{Re} z < 1\}$. Also $\phi_1^{(n)}(0) = \phi_2^{(n)}(0)\ \forall n \geq 0$. From the power series expansion of these two function in $\{z : |z| < 1\}$ it follows that they coincide on this ball, hence on $\{\operatorname{Re} z < 1\}$. In particular they coincide on the imaginary axis which means $\int e^{ist}d\nu_1(t) = \int e^{ist}d\nu_2(t)\ \forall s \in \mathbb{R}$. This is a contradiction.

Problem 2

If K is a compact subset of $\mathbb{R}^n$ show that the set $A = \{x \in \mathbb{R}^n : d(x, K) = 1\}$ has Lebesgue mesure 0.

We first show that if $K \subset B(0, \frac{1}{3})$ then $x \in A \Rightarrow tx \notin A$ for any $t \in (0, \infty)\backslash\{1\}$. Using polar coordinates (c.f. Real and Complex Analysis by Walter Rudin, 3rd Edition, Problem 6, Chapter 8) we conclude from this that A has measure 0. By translation the same conclusion holds if K is contained in *some* open ball of radius $\frac{1}{3}$. The general case is handled by noting that K is the union of a finite number of compact subsets of diameter not exceeding $\frac{1}{3}$ and any point in A has distance 1 from one of these subsets.

Let $K \subset B(0, \frac{1}{3}), d(x, K) = 1$ and $0 < t < 1$. Let $y \in K$ with $\|x - y\| = 1$. Then $\|tx - y\|^2 - \|x - y\|^2 = \|tx - x\|^2 + 2 < tx - x, x - y >= \|tx - x\|^2 + 2(t - 1)\|x\|^2 - 2 < tx - x, y >= (t^2 - 1)\|x\|^2 + 2(1 - t)\|x\|\|y\|$. Now note that $\|y\| < \frac{1}{3}$ and $\|x\| > 1 - \frac{1}{3} = \frac{2}{3}$. Hence $\|tx - y\|^2 - \|x - y\|^2 < 0$ proving that $tx \notin A$. Similarly if $t > 1$ then $\|tx - y\|^2 - \|x - y\|^2 = (t^2 - 1)\|x\|^2 + 2(1 - t)\|x\|\|y\| > \frac{2}{3}(t^2 - 1) - 2(t - 1)\frac{1}{3} > 0$ so $tx \notin A$. This completes the proof.

Problem 3

If $f_n \to 0$ a.e. on a finite measure space $(\Omega, \mathcal{F}, \mu)$ show that there is a sequence $\{a_n\} \uparrow \infty$ such that $a_n f_n \to 0$ a.e.

Solution:

May suppose $f_n \geq 0\ \forall n$. Using $\frac{f_n}{1+f_n}$ we see that we may suppose $0 \leq f_n \leq 1$. Using $\sup\{f_n, f_{n+1}, f_{n+2}, ...\}$ we may suppose $f_n \downarrow 0$. By Egoroff's Theorem we can find a set E_k such that $\mu(E_k^c) < \frac{1}{2^k}$ and integers $n_k \uparrow \infty$ such that $0 \leq f_n < \frac{1}{2^k}$ on E_k for $n \geq n_k$. Let $a_n = 2^{k/2}$ for $n_k < n \leq n_{k+1}$. For $n_k < n \leq n_{k+1}$, $a_n f_n \leq 2^{k/2} f_{n_k} < 2^{-k/2}$ on E_k. Now $\sum_{k=1}^{\infty} \mu(E_k) < \infty$ so almost all points belong to E_k for all k sufficiently large.

Problem 4

Let $x_1, x_2 \in \mathbb{R}^2$. If $A \subset \mathbb{R}^2$ has positive Lebesgue measure show that there exists $y \in \mathbb{R}^2$ and $t \in \mathbb{R}\backslash\{0\}$ such that $y + tx_1$ and $y + tx_2$ both belong to A.

More generally if F is a finite subset of $\mathbb{R}^n$ and $m(A) > 0$ then there exists $y \in \mathbb{R}^2$ and $t \in \mathbb{R}\backslash\{0\}$ such that $y + tx$ belongs to A for all $x \in F$.

Solution: if $x_1 = x_2$ we can take any point u in A and take $y = u - x_1, t = 1$. Let $x_1 \neq x_2$. Let T be a rotation of $\mathbb{R}^2$ such that $T(x_1 - x_2) = \alpha e_1$ where $\alpha = \|x_1 - x_2\|$ and $e_1 = (1, 0)$. Let $B = T(A)$. Not all the sections of B^s are all singletons. $[B^s = \{t : (t, s) \in B]$ because B has positive measure. If (t_1, s) and $(t_2, s) \in B$ with $t_1 \neq t_2$ then $\exists w_1, w_2 \in A$ with $T(w_1) = (t_1, s), T(w_2) = (t_2, s)$ so $w_1 - w_2 = (t_1 - t_2)T^{-1}(e_1) = \frac{1}{\alpha}(t_1 - t_2)(x_1 - x_2)$. Hence $w_1 - \frac{1}{\alpha}(x_1 - x_2) = w_2 - \frac{1}{\alpha}(x_1 - x_2)$. Now take $s = \frac{1}{\alpha}(t_1 - t_2)$ and $y = w_1 - sx_1$. Then $y + sx_1 = w_1 \in A$ and $y + sx_2 = w_1 + s(x_2 - x_1) = w_1 + (w_2 - w_1) = w_2 \in A$, as required.

[A result if Steinhaus in Fund. Mathematica, 1920 says that if A has positive measure in $\mathbb{R}^n$ and F is any finite subset of $\mathbb{R}^n$ then we can find $c \in \mathbb{R}^n$ and $t \in \mathbb{R}$ such that $c + tx \in A$ for any $x \in F$. A special case of this (obtained by taking $n = 1$ and F to be $\{1, 2, 3\}$ is the following: if A is a measurable subset of $\mathbb{R}$ and if $a \in A, b \in A, a \neq b \Rightarrow \frac{a+b}{2} \neq A$ then A has measure 0. Is there a simple proof of this?].

Problem 5

If A and B are subsets of $\mathbb{R}$ of positive measure show that $A + B$ contains an open interval.

Solution: w.o.l.g. assume that the sets are bounded. $I_A \cdot I_B$ is continuous and positive at some point, hence positive in some interval.

Problem 6

If A is a measurable subset of $\mathbb{R}$ such that $a \in A, b \in B, a \neq b \Rightarrow \frac{a+b}{2} \notin A$ then A has measure 0.

Remark: let A be the set of all numbers in $(0, 1)$ whose expansion to base 4 has all coefficients in $\{0, 1\}$. Then A has the property stated here.

Proof: w.l.o.g. A is bounded. Let $\delta = m(A)$. Let f be a continuous function : $\mathbb{R} \to \mathbb{R}$ such that $\displaystyle\int |I_A - f| < \delta/7$. Then $\displaystyle\int |I_A(x)I_A(x+t)I_A(x-t) - f(x)f(x+t)f(x-t)|\, dx < \frac{3\delta}{7}$ and

$I_A(x)I_A(x+t)I_A(x-t) = 0\ \forall x$ if $t \neq 0$. We get $\displaystyle\int f(x)f(x+t)f(x-t)dx \leq \frac{3\delta}{7}$. Let $t \to 0$ to get $\displaystyle\int f^3(x)dx \leq \frac{3\delta}{7}$. This and the inequality $\displaystyle\int |I_A - f| < \delta/10$ give $\displaystyle\int I_A^3(x)dx \leq \frac{6\delta}{7}$ which yields the contradiction $\delta \leq \frac{6\delta}{7}$.

Problem 7 [Steinhaus, 1920, Fund. Math.]

Let A be a measurable subset of $\mathbb{R}$ with positive measure. Let $x_1, x_2, ..., x_k$ be distinct real numbers. Then there exists $c \in \mathbb{R}$ and $t \in \mathbb{R}\backslash\{0\}$ such that $c + tx_i \in A$ for $1 \leq i \leq k$. [Thus, if we are given $d_1, d_2, ..., d_k \in (0, \infty)$ we can fing points in A such that distnces between them are in proportion to $d_1, d_2, ..., d_k$].

Proof: this is similar to the solution of problem 6: just look at $\int |I_A(x+tx_1)I_A(x+tx_2)...I_A(x+tx_k) - f($

Problem 8

Let $f : [a, b] \to \mathbb{R}$. Then f is Lebesgue measurable if and only if the following condition holds: for any $\epsilon > 0$ and any measurable set $A \subset [a, b]$ with $m(A) > 0$ there is a measurable subset B of A such that $m(B) > 0$ and the oscillation of f on B is atmost ϵ.

Proof: if f is measurable then the oscillation of f on $A \cap f^{-1}\{[(i-1)\epsilon, i\epsilon)\}$ is atmost ϵ and this set has positive measure for some i.

Now suppose the given condition holds. For each set A of positive measure and each $\epsilon > 0$ let $\mathcal{F}_{A,\epsilon}$ be the class of all measurable subsets of positive measure contained in A the oscillation of f on which is atmost ϵ. Let $\delta_1 = \sup\{m(B) : B \in \mathcal{F}_{[a,b],\epsilon}\}$. Choose a set B_1 in $\mathcal{F}_{A,\epsilon}$ such that $m(B_1) > \delta_1/2$. If $m(B_1) = b-a$ we stop here. Otherwise we define $\delta_2 = \sup\{m(B) : B \in \mathcal{F}_{B_1^c,\epsilon}\}$ and choose $B_2 \subset B_1^c$ such that $m(B_2) > \delta_2/2$. If $m(B_1 \cup B_2) = b-a$ we stop here. Otherwise we proceed to find $B_3 \subset (B_1 \cup B_2)^c$, etc. We get disjoint measurable sets $B_1, B_2, ...$ such that the oscillation of f on each of these sets is atmost ϵ. Claim: $m(\cup B_n) = b - a$. If this is false then there is a subset E of $(\bigcup_n B_n)^c$ such that $\theta \equiv m(E) > 0$ and the oscillation of f this set is atmost ϵ. Note that there are infintely many $B_n's$ in this case and $\delta_n < 2m(B_n) \to 0$. Hence there is an integer n such that $\delta_n < \theta/2$. Now $E \subset (B_1 \cup B_2 \cup ... \cup B_n)^c$ and the definition of δ_n shows that $\delta_n \geq m(E) = \theta > 2\delta_n$ a contradiction. Now let g_ϵ be $f(x_n)$ on $B_n (n = 1, 2, ...)$ where $x_n \in B_n$ is arbitrary. We get a measurable function g_ϵ with $|f(x) - g_\epsilon(x)| \leq \epsilon$. It follows that f is measurable.

Problem 9

There is no metric d on the set of all Borel measurable maps : $\mathbb{R} \to \mathbb{R}$ such that $f_n \to f$ pointwise if and only if $d(f_n, f) \to 0$.

Proof: $I_Q(x) = \lim_{m\to\infty} \lim_{n\to\infty} [\cos(m!\pi x)]^{2n}$ and there is no sequence from the set $\{[\cos(m!\pi x)]^{2n} : m, n \geq 1\}$ converging pointwise to I_Q. This follows from the fact that pointwise limit of continuous functions is continuous on a dense set. [cf. Hewitt & Stromberg, Exrecise 6.92]

Problem 9

If $(a_n, b_n) \uparrow (a, b)$ and $f \in C^\infty(\mathbb{R})$ is a polynomial on (a_n, b_n) for each n show that f is a polynomial on (a, b).

Proof: if $f = p_n$ on (a_n, b_n) then $p_{n+1} - p_n \equiv 0$ on (a_n, b_n) which implies $p_n \equiv p_{n+1}$. Hence $f = p_1$ on (a, b).

Problem 10

If $f \in C^\infty(\mathbb{R})$ and, for each $x \in \mathbb{R}$ there is an integer $n \geq 0$ such that $f^{(n)}(x) = 0$ then f is a polynomial.

Proof: since $\mathbb{R} = \cup\{x : f^{(n)}(x) = 0\}$ we conclude, from Baire Category Theorem, that f is a polynomial on some open interval. Let U be the union of all open intervals on which f is a polynomial. U is the union of maximal intervals on which f is a polynomial. Such intervals exists by Problem 9. We get disjoint intervals $(a_1, b_1), (a_2, b_2), ...$ We such that f is a polynomial on each of these intervals and their union is U. Let $H = U^c$. Then H is closed and its interior is empty. [if it contains an open interval then a f is a polynomial on a subinterval of the interval, which is a contradiction]. Suppose H has an isolated point a. Then $\exists\ \delta > 0$ such that $[a - \delta, a + \delta] \cap H \subset \{a\}$. On each of the intervals $[a - \delta, a - \frac{1}{k}]$ $(k > \frac{1}{\delta})$, f is a polynomial. [This is because this compact interval is contained in U and hence each point has a neighbourhood on which f is av polynomial]. By Problem 9 f is a polynomial on $[a - \delta, a)$. Similarly, f is a polynomial on $(a, a + \delta]$. This implies, of course, that f is a polynomial on $[a - \delta, a + \delta]$ contradicting the fact that $a \in H$. We have proved that H is a perfect set. Suppose $H \neq \varnothing$. Now $H = \cup\{x \in H : f^{(n)}(x) = 0\}$ and Baire's Theorem shows that there is an integer m and an interior point x_0 of $\{x \in H : f^{(n)}(x) = 0\}$ in H. Hence there is an interval (α, β) such that $x_0 \in H \cap (\alpha, \beta)$ and $H \cap (\alpha, \beta) \subset \{x \in H : f^{(m)}(x) = 0\}$. We claim that f is a polynomial on (α, β). [This would imply that $(\alpha, \beta) \subset U$ contradicting the fact that $H \cap (\alpha, \beta) \neq \varnothing$]. Let $y \in (\alpha, \beta)$. If $y \in H$ then $f^{(m)}(x) = 0$. Otherwise,

y belongs to a maximal interval on which f is a polynomial. Let $y > x_0$. Then the maximal interval (a, b) on which f is a polynomial does not contain x_0 so $x_0 \leq a$. Also a (and b) belong to H by maximality. Note that $f(x) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ on $[a, b]$. We know that $f^{(n)}(a) = 0\ \forall n \geq m$. Indeed, $f^{(n)}(z) = 0\ \forall n \geq m\ \forall z \in H \cap (\alpha, \beta)$ as seen by an induction argument using the fact that each point of $H \cap (\alpha, \beta)$ is a limit of a sequence of distinct points of $H \cap (\alpha, \beta)$. Now $f(x) = \sum\limits_{n=0}^{N-1} \frac{f^{(n)}(a)}{n!}(x - a)^n$ on $[a, b]$. But $y \in (a, b)$ so $f^{(n)}(y) = 0$ $\forall n \geq m$. We have now proved that $f^{(m)}(y) = 0\ \forall y \in H \cap (\alpha, \beta)$ as well as for all $y \in (\alpha, \beta) \backslash H$ and hence f is a polynomial on (α, β). This leads to a contradiction (as already observed) and hence $H = \varnothing$. But then each point of $\mathbb{R}$ has a neighbourhood on which f is a polynomial which implies that f is a polynomial on $\mathbb{R}$. [Use compactness to conclude that f is a polynomial on each of the intervals $[-n, n]$ and then use Problem 9 to complete the proof].

Problem 11

Let (X,d) be a complete metric space and $A \subset X$. Show that there is an equivalent metric on A which makes it complete if and only if A is a G_δ in X.

Proof: if $A = \cap U_n$ with each U_n open in X then $d_1(x,y) = d(x,y) + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\left|\frac{1}{d(x,U_n^c)} - \frac{1}{d(y,U_n^c)}\right|}{1+\left|\frac{1}{d(x,U_n^c)} - \frac{1}{d(y,U_n^c)}\right|}$ defines a metric with desired properties.

Conversely let d_1 be a metric on A which makes it complete and which is equivalent to d. Let $E_n = \{x \in \bar{A} : diam_1(B(x,\delta) \cap A) < \frac{1}{n}$ for some $\delta > 0\}$ where $diam_1$ denotes diameter w.r.t. d_1. It is clear that $A \subset E_n$ $\forall n$. Let $x_0 \in \cap E_n$. For each n $diam_1(B(x,\delta_n) \cap A) < \frac{1}{n}$ for some $\delta_n > 0$. Of course, we can assume that $\delta_n \to 0$. There is a sequence $\{u_j\} \subset A$ such that $u_j \to x_0$ in (X,d). Given $\epsilon > 0$ we can choose n such that $\frac{1}{n} < \epsilon$. Now $d(u_j, x_0) < \delta_n$ and $d(u_k, x_0) < \delta_n$ for j and k sufficiently large and hence $d_1(u_j, u_k) < \epsilon$. Hence $\{u_j\}$ is Cauchy in the complete space (A, d_1). Let $w \in A$ and $d_1(u_j, w) \to 0$. Since d_1 is equivalent to d we get $d(u_j, w) \to 0$ and since $u_j \to x_0$ in (X,d) we get $x_0 = w \in A$. We have now proved that $A = \cap E_n$. If we show that each E_n is open in $\bar{A}$ then we can use the fact the closed set $\bar{A}$ is a G_δ to complete the proof. If $x \in E_n$ then for some $\delta > 0$ $diam_1(B(x,\delta) \cap A) < \frac{1}{n}$. Let $d(u,x) < \delta/2$. Then $diam_1(B(x,\delta/2) \cap A) < \frac{1}{n}$. This proves that E_n is open in $\bar{A}$ for each n.

Problem 13

If $\{a_n\}, \{b_n\}$ are sequences of real numbers such that $a_n \cos(nx) + b_n \sin(nx) \to 0$ as $n \to \infty$ on a set E of positive measure show that $a_n \to 0$ and $b_n \to 0$.

Proof: Let $r_n = (a_n^2 + b_n^2)^{1/2}$ and $(\frac{a_n}{r_n}, \frac{b_n}{r_n}) = (\cos \alpha_n, \sin \alpha_n)$. Then $a_n \cos(nx) + b_n \sin(nx) = r_n \cos(nx - \alpha_n)$. Thus $r_n^2 \cos^2(nx - \alpha_n) \to 0$ $\forall x \in E$. If $r_{n_k} \geq \delta$ for some $\delta > 0$ and sequence $\{n_k\} \uparrow \infty$ then $\int_E \cos^2(n_k x - \alpha_{n_k}) dx \to 0$ which implies $\int_E [1 + \cos(n_k x - \alpha_{n_k})] dx \to 0$. Riemann Lebesgue Lemma now shows that $\int_E dx = 0$, a contradiction. Hence $r_n \to 0$ and $a_n, b_n \to 0$.

Problem 14

If E is a set of finite measure in $\mathbb{R}$ show that $\int_E \cos^{2m}(nx - \alpha_n) dx \to m(E) \frac{1}{2\pi} \binom{2m}{m} 2^{-2m}$ as $n \to \infty$ for any positive integer m and any $\alpha_n' s \in \mathbb{R}$.

Use this to prove the following generalization of Problem 13: $\limsup |a_n \cos(nx) + b_n \sin(nx)| = \limsup [a_n^2 + b_n^2]^{1/2}$ almost everywhere if $\{(a_n, b_n)\}$ is bounded.

Proof: We can write $\cos^{2m} y = c_0 + c_1\cos(2y) + ... + c_m\cos(2my)$ for suitable real numbers $c_0, c_1, ..., c_m$, for all $y \in \mathbb{R}$. This can be seen easily by an induction argument. To compute c_0 we integrate both sides from 0 to 2π. This gives $c_0 = \frac{1}{2\pi}\int_0^{2\pi}(\cos^{2m} y)dy$. Repeated integration by parts gives us $c_0 = \frac{1}{2}\pi\begin{pmatrix} 2m \\ m \end{pmatrix}2^{-2m}$. Now $\int_E \cos^{2m}(nx-\alpha_n)dx = \int_E \sum_{j=0}^{m} c_j\cos(2j(nx-\alpha_n))dy \to c_0 m(E)$ as $n \to \infty$ by Riemann Lebesgue Lemma. This proves the first part. Now let $(a_n, b_n) = (\rho_n\cos t_n, \rho_n\sin t_n)$ (with $\rho_n > 0$). We claim that $\int_E \limsup \rho_n^{2m}\left|\cos(nx-t_n)\right|^{2m}dx \geq \limsup \int_E \rho_n^{2m}\left|\cos(nx-t_n)\right|^{2m}dx$. This follows by applying Fatou's Lemma to $(\sup \rho_n)^{2m} - \rho_n^{2m}\left|\cos(nx-t_n)\right|^{2m}$. Thus $\int_E \limsup \rho_n^{2m}\left|\cos(nx-t_n)\right|^{2m}dx \geq (\limsup \rho_n)^{2m}m(E)\frac{1}{2\pi}\begin{pmatrix} 2m \\ m \end{pmatrix}2^{-2m}$ by the first part. Since this inequality holds for any set $E \subset [0, 2\pi]$ of measure we conclude that $\limsup \rho_n^{2m}\left|\cos(nx-t_n)\right|^{2m} \geq (\limsup \rho_n)^{2m}\frac{1}{2\pi}\begin{pmatrix} 2m \\ m \end{pmatrix}2^{-2m}$ a.e.. Hence $\limsup \rho_n\left|\cos(nx-t_n)\right| \geq (\limsup \rho_n)(\frac{1}{2\pi}\begin{pmatrix} 2m \\ m \end{pmatrix}2^{-2m})^{1/2m}$ a.e. By Stirling's formula we see that $\begin{pmatrix} 2m \\ m \end{pmatrix}2^{-2m} \geq \frac{c}{\sqrt{m}}$ for all m sufficiently large (with $c > 0$). Hence $\limsup \rho_n\left|\cos(nx-t_n)\right| \geq (\limsup \rho_n)(\frac{1}{2\pi}\frac{c}{\sqrt{m}})^{1/2m} \to \limsup \rho_n$ as $m \to \infty$. Since $\rho_n\left|\cos(nx-t_n)\right| \leq \rho_n$ always, the proof is complete.

Problem 15

If $f : \mathbb{R}^2 \to \mathbb{R}$ is separately continuous then it is continuous on a dense set.

Proof: let $f_n(x,y) = f(\frac{i-1}{n}, y) + n(x - \frac{i-1}{n})[f(\frac{i}{n}, y) - f(\frac{i-1}{n}, y)]$ if $\frac{i-1}{n} \leq x \leq \frac{i}{n}$. Then each f_n is continuous on $\mathbb{R}^2$ and $f_n(x,y) \to f(x,y)$ for any $(x,y) \in \mathbb{R}^2$. This implies that f is continuous on a dense set.

Problem 16

Prove or disprove: if $\phi : [0,\infty) \to \mathbb{R}$ is continuous and $\phi(x)p(x) \to 0$ as $x \to \infty$ for every polynomial p then the conclusion of Problem 1 holds with e^{-x} replaced by $\phi(x)$. [i.e. given $f \in C([0,\infty))$ such that $f(x) \to 0$ as $x \to \infty$ and $\epsilon > 0$ there is a polynomial p such that $|f(x) - \phi(x)p(x)| < \epsilon \ \forall x \in [0,\infty)$].

This is false. We have $\int_0^{\infty} e^{-\sqrt[4]{x}}\sin(\sqrt[4]{x})x^n dx = 0$ for $n = 0, 1, 2....$ [See Feller Volume II, p. 224]. Let $d\mu(x) = e^{-\frac{1}{2}\sqrt[4]{x}}\sin(\sqrt[4]{x})dx$ and $\phi(x) = e^{-\frac{1}{2}\sqrt[4]{x}}$. Then

μ is a real measure which integrates every function of the type $\phi(x)p(x)$, where p is a polynomial to, 0. Hence such functions are not dense in the space of continuous functions on $[0,\infty)$ vanishing at ∞ with the supremum norm.

Problem 17

Show that any $\sigma-$ algebra on $\mathbb{N}$ is generated by a finite or countable infinite partition.

Proof: let $\mathcal{F}$be any $\sigma-$ algebra on $\mathbb{N}$. Say $n \sim m$ if, for every $F \in \mathcal{F}$ either n and m both belong to A or both belong to A^c. We claim that the equivalence classes under this equivalence relation form a partition which generates $\mathcal{F}$. Let W be the equivalence class of n. If $m \notin W$ then $\exists\ A_m \in \mathcal{F}$ such that $n \in A_m$ and $m \notin A_m$. We now verify that $W = \bigcap\limits_{m \notin W} A_m$. If $m \notin W$ then $m \notin A_m$. If $m \in W$ then, for any $k \notin W$ we have $n \in A_k$ and $k \notin A_k$. But $m \sim n$ and hence $m \in A_k$. It follows that $m \in \bigcap\limits_{m \notin W} A_m$. This proves that $W = \bigcap\limits_{m \notin W} A_m \in \mathcal{F}$. Thus equivalences classes under $\sim$ form a partition of $\mathbb{N}$ by sets from $\mathcal{F}$. Of course, any partition of $\mathbb{N}$ is necessarily finite or countable infinite. Now let $A \in \mathcal{F}$. We claim that A is the union of all equivalence classes that are contained in A. Let $n \in A$. We have to show that the equivalence class V containing n is a subset of A. If it contains a point $m \in A^c$ then $m \sim n, n \in A$ and $m \in A^c$ which is a contradiction. This finishes the proof.

[Corollary: any measure on any $\sigma-$ algebra on $\mathbb{N}$ extends to a measure on the power set: pick an element from each member of the partition above, call these points $x_1, x_2, ...$ and define $\nu(E) = \sum a_n \delta_{x_n}$ where a_n is the measure of the member of the partition that contains x_n].

Problem 18

If $f : [0,\infty) \to [0,\infty)$ is continuous and if $\sum\limits_{n=1}^{\infty} f(nx) < \infty$ for all $x \geq 0$ show that $\int\limits_0^{\infty} f(x)dx < \infty$. If $\sum\limits_{n=1}^{\infty} f(nx) = \infty$ for all $x \geq 0$ does it follow that $\int\limits_0^{\infty} f(x)dx = \infty$? If $f : [0,\infty) \to [0,\infty)$ is continuous and $\int\limits_0^{\infty} f(x)dx < \infty$ does it follow that $\sum\limits_{n=1}^{\infty} f(n) < \infty$?

Since $[0,\infty) = \bigcup\limits_{N=1}^{\infty} \{x : \sum\limits_{n=1}^{\infty} f(nx) \leq N\}$ we conclude from Baire Category Theorem that there is an integer N and an open interval (a,b) such that

$\sum\limits_{n=1}^{\infty} f(nx) \leq N \ \forall x \in (a,b)$. Hence $\int\limits_a^b \sum\limits_{n=1}^{\infty} f(nx)dx \leq N(b-a)$. This gives $\sum\limits_{n=1}^{\infty} \frac{1}{n} \int\limits_{an}^{bn} f(y)dy < \infty$. This, in turn, implies $\sum\limits_{n=1}^{\infty} \frac{1}{n} \sum\limits_{1+an<j<bn} \int\limits_{j-1}^{j} f(y)dy < \infty$. Since $\sum\limits_{\{n:1+an<j<bn\}}^{\infty} \frac{1}{n} \to \log(\frac{b}{a})$ we are done.

The second assertion is true and the proof is similar.

The third assertion is false:

let $f(n) = 1$, $f(x) = 0$ if $x \notin [n-\alpha_n, n+\alpha_n]$, f "linear" in $[n-\alpha_n, n]$ and $[n, n+\alpha_n]$ where $\alpha_n > 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n < \infty$.

Problem 19

Let $f : [0,\infty) \to [0,\infty)$ is continuous and $f(x+y) - f(x) \to 0$ as $x \to \infty$ for each $y \in [0,\infty)$. Show that the convergence is uniform for y in compact subsets of $[0,\infty)$.

Let $g_n(y) = \sup\{|f(x+y) - f(x)| : x \geq n\}$. Then $\{g_n\}$ is a sequence of bounded measurable functions conveging poitwise to 0. [We remark that if f is uniformly continuous then $g_n's$ are continuous and hence they converge uniformly to 0 on compact sets]. By Egoroff's Theorem there is a set $E \subset [1,2]$ such that $m(E) > 0$ and $g_n \to 0$ uniformly on E. The set $E+E$ contains an interval (a,b) with $0 < a < b < \infty$. We now observe that $g_n(y_1+y_2) \leq g_n(y_1) + g_n(y_2)$ $\forall y_1, y_2 \in [0,\infty)$. Hence $g_n(y) \to 0$ uniformly for $y \in (a,b)$. For $y_1, y_2 \in (a,b)$ with $y_1 > y_2$ we have $|f(x+y_1-y_2) - f(x)| \leq |f(x+y_1-y_2) - f(x-y_2)| + |f(x-y_2) - f(x)| \leq g_m(y_1) + g_m(y_2)$ provided $x - y_2 \geq m$ Note that $x - y_2 \geq m$ if $x \geq b+m$. Let $k = [b+m]+1$. Then $x \geq k \Rightarrow |f(x+y_1-y_2) - f(x)| \leq g_m(y_1) + g_m(y_2)$ so $g_k(y_1 - y_2) \leq g_m(y_1) + g_m(y_2)$. This proves that $g_n \to 0$ uniformly on $(0, b-a)$. Since $g_n(y_1+y_2) \leq g_n(y_1) + g_n(y_2)$ $\forall y_1, y_2 \in [0,\infty)$ we now conclude that $g_n \to 0$ uniformly on compact sets.

Remark1

$g_n \to 0$ uniformly on $[0,\infty)$ if and only if f is a constant. [Indeed $\lim\limits_{t\to\infty} f(t) = f(x)$ $\forall x$ in this case].

Remark 2

Under the hypothesis of this problem, f is necessarily uniformly continuous. [$|f(x+y) - f(x)| < \epsilon$ if $y \in [0,1]$ and $x \geq n$ and n is sufficiently large. Since f is uniformly continous on $[0, n+1]$ it is so on $[0,\infty)$.

Problem 20

Does there exist a non-constant bounded C^∞ function : $\mathbb{R} \to \mathbb{R}$ such that $f^{(n)}(x) \geq 0 \ \forall n \geq 0, \forall x \in \mathbb{R}$?

If yes, give a counter-example. If no, give a real-analytic proof (as opposed to a complex analytic proof).

No. If such an f exists and $a > 0$ then by Taylor's Formula with remainder we see that $f(x) \geq \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$ for $x \geq a, N \in \mathbb{N}$. This shows that $f(x) \to \infty$ as $x \to \infty$ unless . $f^{(n)}(a) = 0$ for $n \geq 1$.

Problem 21

Find a necessary and sufficient condition on a continuous function f on $[0,1]$ under which it can be approximated uniformly by polynomials with integer coefficients.

We claim such an approximation is possible if and only of $f(0)$ and $f(1)$ are integers. 'Only if' part is obvious. For the 'if' part let $g(x) = f(x) - f(0) + [f(0) - f(1)]x$. Then g is continuous and $g(0) = 0 = g(1)$. Let $\{p_k\}$ be the set of all primes (in increasing order) and $h_k(x) = \sum_{j=0}^{p_k} [g(\frac{j}{p_k})p_k] \begin{pmatrix} p_k \\ j \end{pmatrix} \frac{1}{p_k} x^{p_k - j}(1-x)^{p_k}$ where $[f(\frac{j}{p_k})p_k]$ is the greatest integer not exceeding $f(\frac{j}{p_k})p_k$. Since $g(0) = 0 = g(1)$ and $\begin{pmatrix} p_k \\ j \end{pmatrix}$ is an integer multiple of p_k for $1 \leq j \leq p_k - 1$ we see that h_k has *integer* coefficients. Also $\left| h_k(x) - \sum_{j=0}^{p_k} g(\frac{j}{p_k}) \begin{pmatrix} p_k \\ j \end{pmatrix} x^{p_k - j}(1-x)^{p_k} \right| \leq \sum_{j=1}^{p_k - 1} \begin{pmatrix} p_k \\ j \end{pmatrix} \frac{1}{p_k} x^{p_k - j}(1-x)^{p_k} < \frac{1}{p_k}[x + (1-x)]^{p_k} = \frac{1}{p_k}$. Finally we recall that $\sum_{j=0}^{p_k} g(\frac{j}{p_k}) \begin{pmatrix} p_k \\ j \end{pmatrix} x^{p_k - j}(1-x)^{p_k} \to g(x) = f(x) - f(0) + [f(0) - f(1)]x$ uniformly on $[0,1]$.

[What if $[0,1]$ is replaced by a compact interval $[a,b]$?]

Problem 22

If $A \subset \mathbb{R}$ is measurable, $\{x_n\}$ is dense and $x_n + A = A \ \forall n$ show that either $m(A) = 0$ or $m(A^c) = 0$.

If $m(A) > 0$ and $m(A^c) > 0$ then $\exists \ a < b$ such that $(a,b) \subset A - A^c$. Also there is an interger n such that $-x_n \in (a,b)$. But then $-x_n = x - y$ with $x \in A, y \in A^c$ so $y = x + x_n \in x_n + A = A$, a contradiction.

Problem 23

Let $f : [0,1] \to \mathbb{R}$ be a function such that for every $\epsilon > 0$ there is a $\delta > 0$ with $\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \epsilon$ whenever $n \geq 1$ and $\sum_{j=1}^{n} |b_j - a_j| < \delta$. Show that f is Lipschitz. [See Problem 407 for another solution].

Since f is absolutely continuous we can write $f(x) = \int_0^x g(t)dt$ for some integrable function g. Let x be a Lebesgue point of g. Taking $a_j = x, b_j = x + \frac{\delta}{2n}$ for $1 \leq j \leq n$ we get $\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \epsilon$. Hence $n \left| \int_x^{x+\delta/2n} g(t)dt \right| < \epsilon$. This holds for all n and since x is a Lebesgue point of g we get $|g(x)| \leq \frac{2\epsilon}{\delta}$. We have proved that g is an L^∞ function. Hence f is Lipschitz.

Problem 24

Let $a_i < b_i \ \forall i \in I$. Show that $\bigcup_{i \in I} [a_i, b_i]$ can be written as $\bigcup_{n=1}^{\infty} [a_{i_n}, b_{i_n}]$ for some sequence $\{i_n\} \subset I$.

Proof: let $A = \bigcup_{i \in I} [a_i, b_i]$ and $B = \bigcup_{i \in I} (a_i, b_i)$. If $x \in A \backslash B$ then $x \in [a_{i_0}, b_{i_0}]$ for some i_0 and $(a_{i_0}, b_{i_0}) \subset (\alpha, \beta)$ for some component (α, β) of the open set B. Clearly, $x \in [\alpha, \beta] \backslash (\alpha, \beta)$. Hence A is the union of B and an atmost countable set (the end points of the components of B).

Problem 25

Let $a < b$ and $\mathcal{F}$ be a collection of closed non-denerate intervals such that $x \in [a,b]$ implies there exists $\delta > 0$ (possibly depending on x) such that every closed interval of length less than δ containing x belongs to $\mathcal{F}$. Show that there is a partition $\{t_i\}$ of $[a,b]$ such that $[t_{i-1}, t_i] \in \mathcal{F} \ \forall i$.

Let $y = \sup S$ where $S = \{x \in [a,b] : a < t \leq x \Rightarrow$ there is a partition $\{t_i\}$ of $[a,t]$ such that $[t_{i-1}, t_i] \in \mathcal{F} \ \forall i\}$. Clearly $a \in S$. Since $[a, a+\delta] \in \mathcal{F}$ for δ sufficiently small it follows that $y > a$. We claim that $y = b$. If $y < b$ then $[y-\delta, y+\epsilon] \in \mathcal{F}$ for δ and ϵ sufficiently small and there is a point s in $(y-\delta, y]$ that belongs to S. But then $[a, y-\delta]$ has a partition whose sub-intervals are all in $\mathcal{F}$ and $[y-\delta, y+\epsilon] \in \mathcal{F}$ so $y + \epsilon \in S$ for ϵ sufficiently small. This contradicts tha definition of y. Thus $y = b$. Since $[b-\delta, b] \in \mathcal{F}$ for δ sufficiently small it is clear that there is a partition $\{t_i\}$ of $[a,b]$ such that $[t_{i-1}, t_i] \in \mathcal{F} \ \forall i$.

Problem 26

Prove that $[a,b]$ is compact using Problem 25.

Let $\{U_i : i \in I\}$ be an open cover of $[a,b]$. If $x \in [a,b]$ then $x \in U_i$ for some i. Fix such an i for each x. Let $\mathcal{F}_x$ be formed by all closed intervals containing x and contained in U_i and $\mathcal{F}$ be the union of the families $\mathcal{F}_x, x \in [a,b]$. Problem 25 now applies and we get a partition $\{t_j\}$ of $[a,b]$ such that $[t_{j-1}, t_j] \in \mathcal{F}\ \forall j$. But $[t_{j-1}, t_j] \subset U_{i_j}$ for some i_j and the sets U_{i_j} form a finite subcover of $[a,b]$.

Problem 27

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for each real number x there is a $\delta > 0$ with $f(y) \geq f(x)\ \forall y \in (x, x+\delta)$ and $f(y) \leq f(x)\ \forall y \in (x-\delta, x)$. Prove that f is non-decreasing.

For each x consider the collection $\mathcal{F}_x$ of all intervals $[c,d]$ containg x such that $f(y) \geq f(x)\ \forall y \in (x,d]$ and $f(y) \leq f(x)\ \forall y \in [c,x)$. Let $\mathcal{F}$ be the union of the families $\mathcal{F}_x, x \in [a,b]$ where $[a,b]$ is any compact interval in $\mathbb{R}$. By Problem 26 we can find a partition $\{t_j\}$ of $[a,b]$ such that $[t_{j-1}, t_j] \in \mathcal{F}\ \forall j$. In each interval $[t_{j-1}, t_j]$ there is a number x such that $f(t_j) \geq f(x) \geq f(t_{j-1})$. Thus, $f(b) \geq f(a)$. Since a and b are arbitrary points with $a < b$ we are done.

Problem 28

Let $f : [a,b] \to \mathbb{R}$ be differentiable. Show that f is absolutely continuous if and ony if it is of bounded variation.

If f is absolutely continuous then it is of bounded variation, as seen easily from the definition. Let f be differentiable.If f is also of bounded variation then we claim that $f' \in L^1([a,b])$. Once this claim is proved we can apply Theorem 7.21 of Rudin's Real and Complex Analysis (Third Edition) to finish the proof. For proving the claim it suffices to show that the derivative of any monotone function on $[a,b]$ (which exists a.e.) is integrable. Let f be non-decreasing on $[a,b]$. For this note that $\int\limits_0^1 f'(t)dt \leq \liminf \int\limits_0^1 \frac{f(t+h)-f(t)}{h} dt =$ $\liminf\{\frac{1}{h}\int\limits_0^1 f(t+h)dt - \frac{1}{h}\int\limits_0^{1h} f(t)dt\} \leq \liminf f\{(1+h) - f(0)\} = f(1) - f(0)$ where we define $f(t)$ to be $f(1)$ for $t > 1$.

Problem 29

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $F(x) = \sup\{f(x+h) : 0 \leq h \leq \delta\} \in \mathbb{R} \cup \{\infty\}$. Then F has right and left limits at every point.

Proof: If $x_0 < x < x_0 + \delta$ then $[x, x+\delta] = [x, x_0+\delta] \cup [x_0+\delta, x+\delta]$. Hence $F(x) = \max\{\sup\{f(y) : x \leq y \leq x_0+\delta\}, \sup\{f(y) : x_0+\delta \leq y \leq x+\delta\}\}$. This is the maximum of two monotonic functions (one decreasing and the other increasing) and hence $\lim\limits_{x \downarrow x_0} F(x)$ exists. Similarly $\lim\limits_{x \uparrow x_0} F(x)$.

Problem 30

[This is related to Problem 24 above]. Let A be the union of a family of closed balls (of positive radius) in $\mathbb{R}^n$. Is A necessarily a Borel set?

No! Let E be a non-Borel subset of $\mathbb{R}$ and A be the union of the balls $\bar{B}((t,0),1)$ $(t \in E)$ in $\mathbb{R}^2$.Then $A \cap \{\mathbb{R} \times \{1\}\} = E \times \{1\}$. Hence A is not Borel.

Remark: it is known that an arbitrary union of closed balls (of positive radius) in $\mathbb{R}^n$ is Lebesgue measurable.

Problem 31

Prove or disprove that if p is a polynomial of degree n with leading coefficient 1 then $\{x : p(x) > 0, p'(x) > 0, ..., p^{(n)}(x) > 0\}$ is an (open) interval (which may be empty, of course).

True! Note that the result is obvious if $n = 1$. Assume that it is true for polynomials of degree less than n. Then $\{x : p(x) > 0, p'(x) > 0, ..., p^{(n)}(x) > 0\} = \{x : p(x) > 0\} \cap \{x : q(x) > 0, q'(x) > 0, ..., q^{(n-1)}(x) > 0\}$ where $q = p'$. The set $\{x : q(x) > 0, q'(x) > 0, ..., q^{(n-1)}(x) > 0\}$ is an open interval I (by induction hypothesis) on which p is strictly increasing. The set $\{x : p(x) > 0\}$ is the union of a finite number of disjoint intervals (determined by the real zeros of p) and since p is increasing there can be only one of these intervals intersecting $\{x : q(x) > 0, q'(x) > 0, ..., q^{(n-1)}(x) > 0\}$. Hence the result.

Remark: the same argument works for $\{x : p(x) < 0, p'(x) < 0, ..., p^{(n)}(x) < 0\}$. Thus, there is no need to assume that the leading coefficient is 1.

Problem 32

Let $f \in C[0,1]$ and $0 < t_n \downarrow 0$. Suppose there is a constant $C \in (0,\infty)$ such that $|f(x + t_n) - f(x)| \leq Ct_n$ for all n and x with $0 \leq x < x + t_n \leq 1$. Show that f absolutely continous and that it is also of bounded variation. Need f be Lipschitz?

Let $0 \leq a < b \leq 1$. For n so large that $t_n < b - a$ we have the inequalities $|f(x_j + t_n) - f(x_j)| \leq Ct_n$ where $x_j = a + jt_n$ for $j = 0, 2, ..., k_n \equiv [\frac{b-a}{t_n}] - 1$. These inequalities give $\left|f(a) - f(a + t_n[\frac{b-a}{t_n}])\right| \leq C(k_n + 1)t_n$. Letting $n \to \infty$ we get $|f(b) - f(a)| \leq C(b - a)$. We have proved that f is Lipschitz!

Remark: if $\frac{f(x+t_n)-f(x)}{t_n} \to 0$ "boundedly" for some $\{t_n\} \downarrow 0$ and f is continuous then f is a constant.

Problem 32

There is a set $E \subset [0,1]$ of measure 0 such that every Riemann integrable function f on $[0,1]$ has at least one point of continuity in E.

Let $\mathbb{Q} \cap [0,1] = \{r_1, r_2, ...\}$ and $E = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (r_k - \frac{1}{2^{n+k}}, r_k + \frac{1}{2^{n+k}})$. Then E is a dense G_δ of measure 0. Let f be Riemann intergable on $[0,1]$ and D be the points of discontinuity of f. Then D has measure 0 and hence D^c is dense. Further D^c is a G_δ. [It is the set $\bigcap\limits_{n=1}^{\infty}\{x : O_f(x) < \frac{1}{n}\}$]. If we show that the complement of any dense G_δ is of first category it will follow that E^c and D are both of first category. Hence, $E^c \cup D \neq [0,1]$ which means $E \cap D^c \neq \varnothing$, as required. Let V_n be open $\forall n$ and $\bigcap\limits_{n=1}^{\infty} V_n$ be dense. Then $(\bigcap\limits_{n=1}^{\infty} V_n)^c = \bigcup\limits_{n=1}^{\infty} V_n^c$. It remains only to observe that V_n^c is a closed set with no interior.

Problem 33

If $f \in L^1(\mathbb{R})$ and $\int |f(x+y) - f(x)|\, dx = \circ(y)$ as $y \to 0+$ show that $f = 0$ a.e..

Let $s \in \mathbb{R}$. Then $\int e^{isx} f(x+y)dx = e^{-sy}\hat{f}(s)$ and hence $\left|e^{-sy}\hat{f}(s) - \hat{f}(s)\right| = \left|\int e^{isx} f(x+y)dx - \int e^{isx} f(x)dx\right| \leq \int |f(x+y) - f(x)|\, dx = \circ(y)$ as $y \to 0+$. However, $\frac{e^{-ssy}-1}{y} \to -is$ as $y \to 0+$ so $\hat{f}(s) = 0$ for all $s \neq 0$. Since $\hat{f}$ is continuous we get $\hat{f}(s) = 0$ for all s which implies $f = 0$ a.e.

Second proof: let a and b be Lebesgue points of f with $a < b$. Then $\frac{1}{y}\int\limits_a^{a+y} f(t)dt - \frac{1}{y}\int\limits_b^{b+y} f(t)dt = -\frac{1}{y}\int\limits_{a+y}^{b+y} f(t)d + \frac{1}{y}\int\limits_a^{b} f(t)dt = \frac{1}{y}\int\limits_a^{b} f(t)dt - \frac{1}{y}\int\limits_a^{b} f(t+y)dt \to 0$ by hypothesis. Hence $f(b) = f(a)$. This proves that f is a.e. constant and the constant must be 0 by integrability.

Problem 34

Let μ be a finite positive measure (or a complex measure) on the Borel $\sigma-$field of $\mathbb{R}$. Let $0 < c < 1$ and suppose $m(A) = c \Rightarrow \mu(A) = 0$ (where m is the Lebesgue measure). Show that $\mu = 0$.

We have $\mu([x, x+c]) = 0$ $\forall x$. Integrating w.r.t x from $-\infty$ to b we get $\int\limits_{-\infty}^{b}\int I_{[x,x+c]}(y)d\mu(y)dx = 0$. By Fubini's Theorem this gives $\int\int\limits_{-\infty}^{b} I_{[y-c,y]}(x)dxd\mu(y) = 0$. This means $\int\{\min\{y,b\}-y+c\}d\mu(y) = 0$ $\forall b$. If $b_1 < b_2$ we get $\int\{\min\{y,b_2\}-\min\{y,b_1\}\}d\mu(y) = 0$. Thus $\int\limits_{b_1}^{b_2}(y-b_2)d\mu(y) + \int\limits_{b_2}^{\infty}(b_2-b_1)d\mu(y) = 0$. Let x be any real number such that $\lim\limits_{\delta\to 0}\frac{\mu(x-\delta,x+\delta)}{2\delta}$ exists (and is finite). Taking $b_1 = x-\delta$ and

$b_2 = x + \delta$ in above equation, dividing the equation by 2δ and letting $\delta \to 0$ we get $\mu(x, \infty) = 0$. This holds almost everywhere and hence everywhere (because $\mu(x, \infty)$ is a right-continuous function). This implies that $\mu(A) = 0$ for every Borel set A.

Problem 35

Show that any $f \in C[0,1]$ can be written as $g + h$ where g and $h \in C[0,1]$ and they are both nowhere differentiable.

Let $S = \{f - \phi : \phi \in C[0,1]$ and ϕ is nowhere differentiable$\}$. Since the complement of the set of nowhere difrferentiable functions is of first category we see that S^c is of first category in $C[0,1]$. This implies that there is atleast one nowhere differentiable function in S. Let g be such a function Then g and $f - g$ are both nowhere differentiable.

Remark. Similarly we can show that any bounded measurable function on $\mathbb{R}$ is the sum of two bounded measurable functions each of which is one-to-one.

Problem 36

Construct a topological space (X, τ) and a sequence of measurable functions $\{f_n\}$ from $[0,1]$ into X such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists $\forall x \in [0,1]$ but f is not measurable. [Here measurability is w.r.t. the Borel $\sigma-$ fields on $[0,1]$ and X].

Let $f_n(x)(t) = \max\{0, 1 - n\,|x - t|\}$. If $E \subset [0,1]$ is non-Borel then $V = \cup\{\{f : f(t) > 1/2\} : t \in E\}$ is open in $X = [0,1]^{[0,1]}$ with the product topology and $I_{\{x\}}$, which is the pointwise limit of $\{f_n\}$, is not measurable because the inverse image of V is precisely E.

Remark

If X is a metric space and $\{f_n\}$ is a sequence of measurable functions $\{f_n\}$ from $[0,1]$ into X such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists $\forall x \in [0,1]$ then f is measurable. To see this let U be open in X and $g(x) = d(x, U^c)$. Then g is continuous and hence $g \circ f_n$ is measurable for each n. Hence $g \circ f$ is also measurable. Now $\{x : g(f(x)) = 0\}$= $\{x : f(x) \in U^c\}$ and hence $f^{-1}(U)$ is a Borel set in $[0,1]$.

Problem 37

Let H be a complex Hilbert space and $T : H \to H$ an isometry which is not onto. Show that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

The range of T is closed so there is a non-zero vector y in $T(H)^{\perp}$. Let $|\lambda| < 1$. Then $< T^*y, x >=< y, Tx >= 0$ $\forall x$. Hence $T^*y = 0$. We claim that λ is an eigen value of T^* with eigen vector $\sum_{k=0}^{\infty} \lambda^k T^k y$. First note that the series $\sum_{k=0}^{\infty} \lambda^k T^k y$ is convergent because $|\lambda| \|T\| \leq |\lambda| < 1$. Also $\sum_{k=0}^{\infty} \lambda^k T^k y \neq 0$ because the sequence $\{T^k y\}$ is orthogonal: $< T^k y, T^j y >=< y, (T^k)^* T^j y >=< y, T^{j-k} y >= 0$ if $j > k$. Since $T^*(\sum_{k=0}^{\infty} \lambda^k T^k y) = T^* y + \sum_{k=1}^{\infty} \lambda^k T^{k-1} y = \lambda \sum_{k=0}^{\infty} \lambda^k T^k y$ we conclude that λ is an eigen value of T^* and hence that $\bar{\lambda} \in \sigma(T)$. We have proved that $\{\lambda : |\lambda| < 1\} \subset \sigma(T)$ which completes the proof since $\sigma(T) \subset \{\lambda : |\lambda| \leq 1\}$.

Problem 38

Let H be a Hilbert space and P, Q be projections on M and N respectively. Prove that $\{(PQ)^n x\}$ converges for every x. What can you say about the operator $\lim_{n\to\infty} (PQ)^n$?

We show that the (pointwise) limit is the projection on $M \cap N$. Let $A_n = (QP)^{n/2}$ or $P(QP)^{(n-1)/2}$ according as n is even or odd.

If $n + m$ is odd then $< A_n x, A_m y >=< A_{n+m} x, x >$. Similarly we see that if n and m are even then $< A_m x, A_n x >=< A_{n+m-1} x, x >$ and if if n and m are odd then $< A_m x, A_n x >=< x, A_{n+m-1} x >$ .. Now $\|A_m x - A_n x\|^2 =< A_m x, A_m x > + < A_n x, A_n x > -2 \operatorname{Re} < A_m x, A_n x >$. Using above identities we see that if we can show that $< A_{2j-1} x, x >$ has a limit in $\mathbb{R}$ as $j \to \infty$ we can conclude that $\{A_n x\}$ is Cauchy.In particular we see that $\lim_{n\to\infty} A_{2n} x$ exists which means $Ax \equiv \lim_{n\to\infty} (QP)^n x$ exists. Now $PA_{2n} = A_{2n+1}$ and $QA_{2n-1} A_{2n}$ so we get $PA = A = QA$. Any point in the range of A is a fixed point for both P and Q and hence $range(A) \subset N \cap M$. But on $N \cap M$ it is obviuous that $Ax = x$ and hence $N \cap M \subset range(A)$. Thus $range(A) = N \cap M$ and $A = I$ on $range(A)$. Next we observe that $PA = A$ and $QA = A$. These follow from the realtions $PA_{2n} = A_{2n+1}$ and $QA_{2n-1} = A_{2n}$. Thus, $A^* = A^* P$ and $A^* = A^* Q$ which means A^* vanishes on the ranges of $(I - P)$ and $(I - Q)$ which are $M^\perp$ and $N^\perp$. Hence $A^* = 0$ on $M^\perp + N^\perp = (N \cap M)^\perp$. From the fact that $A = I$ on $N \cap M$ it follows that $A^* = I$ on $N \cap M$. We have proved that A^* is the projection on $N \cap M$. It follows that A^* is self-adjoint and this implies that A is also self-adjoint. Since $A^2 = A$ we conclude that A is the projection onti its range which is $N \cap M$.

It remains to show that $< A_{2j-1} x, x >$ has a limit (in $\mathbb{R}$) as $j \to \infty$. We prove that the sequence $\{< A_{2j-1} x, x >\}$ is actually non-negative and decreasing. Since $< A_{2j-1} x, x >=< A_j x, A_j x >$ it follows that the sequence is non-negative. Next we note that $A_{j+1} = PA_j$ if j is even and $A_{j+1} = QA_j$ if j is odd. It follows that $\|A_{j+1} x\|^2 \leq \|A_j x\|^2$ which proves that the sequence $\{< A_{2j-1} x, x >\}$ is decreasing.

We have proved that $(QP)^n = A_{2n} \to A$, the projection on $N \cap M$. By symmetry it follows that $(PQ)^n$ also converges to the projection on $N \cap M$.

Remark: let X, Y be independent random variables and Z have finite second moment. Let $Z_1 = E(Z|X), Z_2 = E(Z_1|Y), ..., X_{2n} = E(X_{2n-1}|Y), X_{2n+1} = EX_{2n}|X), ...$ Then $Z_n \to EZ$ in the mean. This follows immediately from above result and the fact that if a random variable is measurable w.r.t. the sigma filed genertaed by X as well as the sigma filed genertaed by Y then it is independent of itself and hence a.s. constant.

Problem 39

Let M be a closed linear subspace of $L^1[0,1]$ such that $M \subset \bigcup_{p>1} L^p[0,1]$. Show that $M \subset L^p[0,1]$ for some $p > 1$.

Since $M \subset \bigcup_{p>1, N \geq 1} \{f \in S : \int |f|^p \leq N\}$ and since $\{f \in S : \int |f|^p \leq N\}$ is closed in M we can find (by Baire-Category Theorem) $p > 1$ and $N \in \mathbb{N}$ such that $\{f \in S : \int |f|^p \leq N\}$ contains an open ball $B(f_0, \delta)$ in S. It follows that if $f \in S$ then $f_0 + \frac{\delta}{2} \frac{f}{\|f\|_1} \in \{f \in S : \int |f|^p \leq N\}$ and $f_0 \in \{f \in S : \int |f|^p \leq N\}$ so $\frac{\delta}{2} \frac{f}{\|f\|_1} \in \{f \in S : \int |f|^p \leq N\}$. Thus $\int |f|^p < \infty \; \forall f \in S$.

Problem 40

Prove or disprove: if $k \in \mathbb{N}$ and $\{p_n\}$ is a sequence of polynomials of degree not exceeding k converging pointwise to 0 on $[0,1]$ then $p_n \to 0$ uniformly.

True. Consider the statement: $\{p_n\}$ is a sequence of polynomials of degree not exceeding k converging pointwise to 0 on $[0, \delta]$ for some $\delta > 0$ then $p_n \to 0$ uniformly.

We prove the validity of this statement by induction on k. For $k = 1$ the proof is trivial.Assume that it holds for a certain k. Let $p_n(x) = \sum_{j=0}^{k+1} a_{n,j} x^j \to 0$ pointwise on $[0, \delta]$. Then $a_{n,0} \to 0$ so $\sum_{j=1}^{k+1} a_{n,j} x^j \to 0 \; \forall x \in [0, \delta]$. Hence $\sum_{j=1}^{k+1} a_{n,j}(x^j - \delta^j) \to 0 \; \forall x \in [0, \delta]$. This gives $a_{n,1} + a_{n,2}(x + \delta) + a_{n,3}(x^2 + x\delta + \delta^2) + ... + a_{n,(k+1)}(x^k + x^{k-1}\delta + ... + x\delta^{k-1} + \delta^k) \to 0$ if $0 \leq x < \delta$. By induction hypothesis this gives $a_{n,k+1} \to 0, a_{n,k} + a_{n,k+1}\delta \to 0, ..., a_{n,1} + a_{n,2}\delta + a_{n,3}\delta^2 + ... + a_{n,(k+1)}\delta^k \to 0$. Clearly these imply that $a_{n,j} \to 0$ for each j. This completes the induction argument.

Remark

There exist sequence of polynomials on $\mathbb{C}$ converging pointwise to a discontinuous function. For example if $\{P_n\}$ is the sequence constructed in Example 8.15 (page 264) of "An Introduction to Classical Complex Analysis" by Robert B. Burckel then $\operatorname{Re} P_n(ix) \to 0$ for $x \in \mathbb{R}\backslash\{0\}$ and $\operatorname{Re} P_n(ix) \to 1$ for $x = 0$. Let $p_n(x) = \operatorname{Re} P_n(ix), 0 \leq x \leq 1$. Then $\{p_n\}$ converges pointwise to a discontinuous function. Since the convergence is not uniform we can find $\epsilon > 0$ and an increasing sequence $\{n_j\}$ of positive integers such that $\sup\{\left|p_{n_k}(x) - p_{n_{k+!}}(x)\right| \geq \epsilon$ for each k. It follows that the sequence of polynomials $\{p_{n_k} - p_{n_{k+!}}\}$ converges to 0 pointwise, but not uniformly. Thus, the hypotheses that the degrees of polynomials p_n are bounded cannot be omitted from Problem 40.

Problem 41

Let (X, d) be a metric space such that every decreasing sequence of closed sets with diameters approaching 0 has non-empty intersection. Can we conclude that (X, d) is complete?

Yes! Let $\{x_n\}$ be Cauchy. There is a subsequence $\{n_j\}$ of $\{1, 2, ...\}$ such that $d(x_{n_j}, x_{n_{j+1}}) \leq \frac{1}{2^j}$ $\forall j$. Consider the closed balls C_j with center x_{n_j} and radius $\frac{1}{2^{j-1}}$. If $x \in C_{j+1}$ then $d(x, x_{n_{j+1}}) \leq \frac{1}{2^j}$ and $d(x_{n_j}, x_{n_{j+1}}) \leq \frac{1}{2^j}$. Hence $d(x, x_{n_j}) \leq \frac{1}{2^j} + \frac{1}{2^j} = \frac{1}{2^{j-1}}$ which means $x \in C_j$. By hypothesis there is a point x in $\bigcap\limits_{j=1}^{\infty} C_j$. Clearly, $x_{n_j} \to x$. Since $\{x_n\}$ is Cauchy this implies that $x_n \to x$.

Problem 42

Let $f : [0, 1] \to \mathbb{R}$ be continuous and non-decreasing. Show that there is a sequence of polynomials $\{p_n\}$ such that $p_n \uparrow f$ uniformly on $[0, 1]$ and each p_n is non-decreasing.

Extend f to $\mathbb{R}$ so that the extended f is uniformly continuous, non-decreasing and bounded on $\mathbb{R}$. Let $g_t(x) = \sqrt{\frac{t}{\pi}} \int f(x - y)e^{-ty^2}dy$ where $t > 0$. Then g is non-decreasing and continuously differentiable. Also, $g_t(x) \to f(x)$ uniformly for $0 \leq x \leq 1$ as $t \to 0$. We claim that for each t, there is a sequence of polynomials $\{p_n\}$ such that $p_n \to g_t$ uniformly on $[0, 1]$ and each p_n is non-decreasing. Let $\{q_n\}$ be a sequence of *non-negative* polynomials converging to g_t' uniformly on $[0, 1]$. [To see that this is possible just approximare $\sqrt{g_t'}$ by polynomials ϕ_n and take $q_n = \phi_n^2$]. Let $p_n(y) = g_t(0) + \int\limits_0^y q_n(s)ds$. Then $p_n \to g_t$ uniformly on $[0, 1]$ and each p_n is non-decreasing.. This proves our claim. Finally we show that we can modify $\{p_n\}$ so that $p_n(x) \leq p_{n+1}(x)$ $\forall x$, $\forall n$. Applying the result just proved to $f - \frac{1}{2^n}$ in place of f we get a non-decreasing polynomial ξ_n such that $\left|\xi_n(x) - (f(x) - \frac{1}{2^n})\right| < \frac{1}{2^{n+2}}$ $\forall x$, $\forall n$. Then

$\xi_n(x) < f(x) - \frac{1}{2^n} + \frac{1}{2^{n+2}} < f(x) - \frac{1}{2^{n+1}} - \frac{1}{2^{n+3}} < \xi_{n+1}(x)$ and $|\xi_n(x) - f(x)| < \frac{1}{2^n} + \frac{1}{2^{n+2}}\ \forall x,\ \forall n.$

Problem 43

Let $f : [0,1] \to \mathbb{R}$ be continuous and one-to-one. Show that there is a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly on $[0,1]$ and each p_n is one-to-one.

f is strictly increasing and we find a striclty increasing sequence of strictly increasing polynomials $\{\xi_n\}$ converging uniformly by the argument of Problem 42.

Problem 44

If P, Q and PQ are projections on a Hilbert space and $P \neq Q$ show that $\|P - Q\| = 1$.

Since PQ is self adjoint we have $PQ = QP$. Note that $(P - Q)^3 = P - Q$ and hence $(P - Q)^{3^n} = P - Q$ for any positive integer n. It follows that $\|P - Q\| \leq \|P - Q\|^{3^n}$ which implies $\|P - Q\| \geq 1$ since $P \neq Q$. Now $\|Px - Qx\|^2 = \|P(I - Q)x - (I - P)Qx\|^2 = \|P(I - Q)x\|^2 + \|(I - P)Qx\|^2$ (because the ranges of P and $(I-P)$ are orthogonal) and this gives $\|Px - Qx\|^2 \leq \|P\|^2 \|(I - Q)x\|^2 + \|I - P\|^2 \|Qx\|^2 \leq \|(I - Q)x\|^2 + \|Qx\|^2 = \|x\|^2$.

Remark: we actually have $(P - Q)^n = P - Q$ for any odd positive integer $n \geq 3$. To see this note that $(P - Q)^n = \sum_{j=1}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix} PQ + P + (-1)^n Q$ and that $0 = (1-1)^n = \sum_{j=1}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix} + 1 + (-1)^n$. Thus $(P - Q)^n = -[1 + (-1)^n]PQ + P + (-1)^n Q = P - Q$ if n is odd.

Problem 45

Let p be q be polynomials with real coefficients. Show that if $\max\{p(x), q(x)\}$ is a polynomial then either $p(x) \leq q(x)\ \forall x$ or $q(x) \leq p(x)\ \forall x$. Show that the same conclusion holds if $\min\{p(x), q(x)\}$ is a polynomial.

Suppose p and $|p|$ are polynomials. If p has a real root x_0 then $p(x) = (x - x_0)^k \phi(x)$ for some polynomial ϕ with $\phi(x_0) \neq 0$ and some positive integer k. Since ϕ does not change sign near x_0 it follows that $|x - x_0|^k$ is infinitely differentiable at x_0. But $|x - x_0|^k$ does not have a $k-th$ derivative at x_0. Thus p does not have any real roots, so either $p(x) \geq 0\ \forall x$ or $p(x) \leq 0\ \forall x$. We can now complete the proof using the identity $|p - q| = 2\max\{p, q\} - p - q$.

Problem 46

Find a necessary and sufficient condition on a sequence $\{b_n\}$ of real numbers that $\sum a_n b_n$ converges whenever $\sum a_n$ converges.

The condition is $\sum |b_n - b_{n+1}| < \infty$. If this condition holds then $\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} (s_n - s_{n-1}) b_n = \sum_{n=1}^{N} s_n (b_n - b_{n+1}) + s_N b_{N+1}$ where $s_0 = 0$ and $s_n = a_1 + a_2 + ... + a_n (n \geq 1)$. Since the series $\sum_{n=1}^{\infty} s_n (b_n - b_{n+1})$ is absolutely convergent, $\{s_n\}$ is convergent and $\{b_n\} = \{b_1 + (b_2 - b_1) + ... + (b_n - b_{n-1})\}$ is also convergent we see that $\sum a_n b_n$ converges.

Now suppose $\sum a_n b_n$ converges whenever $\sum a_n$ converges. First note that $\{a_n b_n^+\}$ and $\{a_n b_n^-\} \in l^1$ whenever $\{a_n\} \in l^1$. [if $c_n = |a_n|$ if $b_n \geq 0$ and 0 otherwise then $\sum c_n b_n$ converges by hypothesis and this implies $\{a_n b_n^+\} \in l^1$. Similarly $\{a_n b_n^-\} \in l^1$]. By a standard argument using Uniform Boundedness Principle we get $\{b_n^+\}$ and $\{b_n^-\} \in l^\infty$. Hence $\{b_n\}$ is bounded. We now consider the space c of all convergent sequences with the supremum norm. Define $T_N : c \to \mathbb{C}$ by $T_N\{s_n\} = \sum_{n=1}^{N} s_n (b_n - b_{n+1}) + s_N b_{N+1}$. This is a sequence of continuous linear functionals on c and we claim that $\lim_{N \to \infty} T_N\{s_n\}$ exists for every sequence $\{s_n\} \in c$. To see this write a_n for $s_n - s_{n-1}$ $(s_0 = 0)$. Then $T_N\{s_n\} = \sum_{n=1}^{N} a_n b_n$. The claim now follows from the fact that $\sum a_n$ converges. By Uniform Boundedness Principle there is a constant $M \in (0, \infty)$ such that $\left| \sum_{n=1}^{N} s_n (b_n - b_{n+1}) + s_N b_{N+1} \right| \leq M \; \forall N$ and for all sequences $\{s_n\}$ with $|s_n| \leq 1$ $\forall n$. Since $\{b_n\}$ is bounded it follows that $\left| \sum_{n=1}^{N} s_n (b_n - b_{n+1}) \right| \leq M + \sup |b_n|$. By an appropriate choice of $\{s_1, s_2, ..., s_N\}$ we conclude that $\sum_{n=1}^{N} |b_n - b_{n+1}| \leq M + \sup |b_n| \; \forall N$.

Problem 47

Consider the colection of all polynomials on $[0, 1]$ with the ordering $p \leq q$ if $p(x) \leq q(x) \; \forall x$. Let p and q be any two polynomials. Show that one of the following is true:

a) $p(x) \leq q(x) \; \forall x$ or $q(x) \leq p(x) \; \forall x$

b) there is no smallest polynomial ϕ exceeding both p and q

Let $f(x) = \max\{p(x), q(x)\}$. Then f is continuous but it is not a polynomial. [See Problem 45 above]. There is a sequence of polynomials $\{h_n\}$ *decreasing* uniformly to f. [See the last part of the solution to Problem 42 above]. If there is a smallest polynomial ϕ exceeding both p and q then $\phi \leq h_n \ \forall n$ and hence $\phi \leq f$. But $\phi \geq p$ and $\phi \geq q$ so $\phi \geq f$. Thus $f = \phi$ is a polynomial which is a contradiction.

Problem 48

Show that if T and S are commuting operators on a normed linear space then $\rho(T+S) \leq \rho(T) + \rho(S)$ where $\rho(T) = \limsup \|T^n\|^{1/n}$ (the spectral radius of T). Give examples of 2×2 matrices A and B such that $\rho(A+B) > \rho(A)+\rho(B)$.

Since $\|(T+S)^n\|^{1/n} \leq \sum\limits_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} \|T^j\| \, \|S^{n-j}\|$ we only have to show that $\limsup(\sum\limits_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} \alpha_j \beta_{n-j})^n \leq \limsup(\alpha_j)^{1/j} + \limsup(\beta_j)^{1/j}$. This is easy. Let $A = \begin{matrix} 0 & 4 \\ 1 & 0 \end{matrix}$ and $B = \begin{matrix} 0 & 1 \\ 4 & 0 \end{matrix}$. Then $\rho(A) = \rho(B) = 2$ and $\rho(A+B) = 5 > 2+2$.

Problem 49

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, integrable and of bounded variation. Show that $\sum\limits_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{2\pi} \sum\limits_{n=-\infty}^{\infty} \hat{f}(n)$.

Let $V(x)$ be the variation of f on $(-\infty, x]$. Let $g(x) = \sum\limits_{n=-\infty}^{\infty} f(x+2\pi n), x \in \mathbb{R}$. We claim that g is well defined, continuous, of period 2π and of bounded variation on $[0, 2\pi]$. Once this claim is established we can conclude that the Fourier series of g converges to g at every point. In particular, $\sum\limits_{n=-\infty}^{\infty} f(2\pi n) = g(0) = \sum\limits_{n=-\infty}^{\infty} \hat{g}(n)$ and since $\hat{g}(k) = \frac{1}{2\pi} \int\limits_0^{2\pi} g(x) e^{-ikx} dx$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int\limits_0^{2\pi} f(x+2\pi n) e^{-ikx} dx = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int\limits_{2\pi n}^{2\pi(n+1)} f(x) e^{-ikx} dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}(k).$$

The interchange of the sum and the integral here follows by uniform convergence of the series (to be eastablished). Note that we are using the notation $\hat{h}$ in two senses: for an function $h \in L^1(\mathbb{R})$, $\hat{h}(t) = \int\limits_{-\infty}^{\infty} h(x)e^{-itx}dx$ whereas for a periodic function on $[0, 2\pi]$ with period 2π it is $\frac{1}{2\pi}\int\limits_{0}^{2\pi} h(x)e^{-ikx}dx$.

We now prove the claim. Note that for any interval $[a, b]$ of legth 2π, $\int\limits_{a}^{b} \sum\limits_{n=-\infty}^{\infty} |f(x+2\pi n)|\,dx = \sum\limits_{n=-\infty}^{\infty} \int\limits_{a}^{b} |f(x+2\pi n)|\,dx = \sum\limits_{n=-\infty}^{\infty} \int\limits_{a+2\pi n}^{b+2\pi n} |f(x)|\,dx = \int\limits_{-\infty}^{\infty} |f(x)|\,dx < \infty$. Hence g is well defined almost everywhere on $\mathbb{R}$. Let $x < y < x + 2\pi$. Then $|f(y+2\pi n) - f(x+2\pi n)| \leq V(y+2\pi n) - V(x+2\pi n) \leq V(x+2\pi(n+1)) - V(x+2\pi n)$. The series $\sum\limits_{n=-\infty}^{\infty} [V(x+2\pi(n+1)) - V(x+2\pi n)]$ is convergent. It follows easily from this that g is continuous on $[x, x+2\pi]$ for any real number x. Thus g is continuous on $\mathbb{R}$. It remains only to show that g is of bounded variation on $[0, 2\pi]$. If $\{t_j\}_{0 \leq j \leq N}$ is a partition of $[0, 2\pi]$ then $\sum\limits_{j=0}^{N} |g(t_j) - g(t_{j-1})| \leq \sum\limits_{j=0}^{N} \sum\limits_{n=-\infty}^{\infty} |f(t_j+2\pi n) - f(t_{j-1}+2\pi n)| = \sum\limits_{n=-\infty}^{\infty} \sum\limits_{j=0}^{N} |f(t_j+2\pi n) - f(t_{j-1}+2\pi n)| \leq \sum\limits_{n=-\infty}^{\infty} [V(2\pi(n+1)) - V(2\pi n)] < \infty$.

Problem 50

Let $\{f_n\}$ be an orthonormal basis of $L^2([0, 2\pi])$. Show that $\sum\limits_{n=-\infty}^{\infty} \int |f_n(x)|\,dx = \infty$.

There is a function $g \in L^2([0, 2\pi])$ such that $\hat{g}(n) = \frac{1}{n}$ $\forall n \in \mathbb{N}$. Note that $\sum\limits_{j=-\infty}^{\infty} \left|\hat{f_j}(n)\right|^2 = \sum\limits_{j=-\infty}^{\infty} \left|< f_j, e^{inx} >\right|^2 = \left\|e^{inx}\right\|_2^2 = 1$ $\forall n \in \mathbb{N}$. Now $\infty = \sum\limits_{n=-\infty}^{\infty} \frac{1}{n} = \sum\limits_{n=-\infty}^{\infty} \frac{1}{n} \sum\limits_{j=-\infty}^{\infty} \left|\hat{f_j}(n)\right|^2$

$= \sum\limits_{j=-\infty}^{\infty} \sum\limits_{n=-\infty}^{\infty} \hat{g}(n) \left|\hat{f_j}(n)\right|^2 = \sum\limits_{j=-\infty}^{\infty} \sum\limits_{n=-\infty}^{\infty} \hat{g}(n)\hat{f_j}(n)[\hat{f_j}(n)]^- = \sum\limits_{j=-\infty}^{\infty} < g * f_j, f_j >= \sum\limits_{j=-\infty}^{\infty} < g, f_j *, \tilde{f_j} >$

$\leq \sum\limits_{j=-\infty}^{\infty} \|g\|_2 \left\|f_j *, \tilde{f_j}\right\|_2 \leq \sum\limits_{j=-\infty}^{\infty} \|g\|_2 \|f_j\|_2 \|f_j\|_1 = \|g\|_2 \sum\limits_{j=-\infty}^{\infty} \|f_j\|_1$.

Problem 51

Construct probability measures $\mu_n, \nu_n, n \geq 1$ on $[0,1]$ such that $\int f d\mu_n - \int f d\nu_n \to 0$ for every continuous function $f : [0,1] \to \mathbb{R}$ but $\mu_n([0,x]) - \nu_n([0,x]) \nrightarrow 0$ for any $x \in [0,1)$.

Let $[a_1, b_1], [a_2, b_2], ...$ be the intervals $[0,1/2], [1/2,1], [0,1/2^2], [1/2^2,1/2], [1/2,3/2^2], [3/2^2,1], ...$. Let $\mu_n = \delta_{b_n}$ and $\nu_n = \delta_{a_n}$. If $f : [0,1] \to \mathbb{R}$ is continuous then it is unifrmly continuous and $b_n - a_n \to 0$ so $\int f d\mu_n - \int f d\nu_n = f(b_n) - f(a_n) \to 0$. If $x \in (0,1)$ then $\mu_n([0,x]) - \nu_n([0,x])$ takes the values 0 and -1 each for infinitely many n. For $x = 0, \mu_n([0,x]) - \nu_n([0,x])$ take the values 1 and 0 each for infintely many n. .

Problem 52

Let $(\Omega, \Im, P)$ be a probability space and $X, X_1, X_2, ...$ be random variables on it. Show that $X_n \xrightarrow{P} X$ if and only if $Q \circ X_n^{-1} \xrightarrow{w} Q \circ X^{-1}$ for every probablility measure Q on $(\Omega, \Im)$ which is equivalent to P (in the sense $P << Q$ and $Q << P$)

We first assume that $X, X_1, X_2, ..$ are uniformly bounded. If $X_n \xrightarrow{P} X$ and $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function then $\int f d(Q \circ X_n^{-1}) \to \int f d(Q \circ X_n^{-1})$ because $\int f(X_n)) \frac{dQ}{dP} dP \to \int f(X)) \frac{dQ}{dP} dP$ by Dominated Convergence Theorem. Conversely suppose $Q \circ X_n^{-1} \xrightarrow{w} Q \circ X^{-1}$ for every probablility measure Q on $(\Omega, \Im)$ which is equivalent to P. We claim that $\int\limits_A X_n dP \to \int\limits_A X dP$ for any $A \in \Im$. Once this is proved we see that $\|X_n\|_2 \to \|X_2\|$ and $< X_n, Y > \to < X, Y >$ for any simple function Y. These facts imply that $\|X_n - X\|_2 \to 0$ and hence $X_n \xrightarrow{P} X$. To prove the calim we assume that $P(A) > 0$ and $\left| \int\limits_A X_{n_k} dP - \int\limits_A X dP \right| \geq \epsilon, k = 1, 2, ...$ for some $\epsilon > 0$ and some $n_1 < n_2 ....$ Let $Q(E) = \frac{(1-\epsilon)P(E \cap A) + \epsilon P(E \cap A^c)}{(1-\epsilon)P(A) + \epsilon P(A^c)}$. Then Q is equivalent to P so $(1-\epsilon) \int\limits_A X_n dP + \epsilon \int\limits_{A^c} X_n dP \to (1-\epsilon) \int\limits_A X dP + \epsilon \int\limits_{A^c} X dP$. But $\int X_n dP \to \int X dP$ so $(1 - 2\epsilon) \int\limits_A X_n dP \to (1 - 2\epsilon) \int\limits_A X dP$.

For the general case we note that $X_n \xrightarrow{P} X$ if and only if $\tan_n^{-1} X_n \xrightarrow{P} \tan^{-1} X$ and $\int fd(Q \circ X_n^{-1}) \to \int fd(Q \circ X_n^{-1})$ if and only if $\int fd(Q \circ (\tan^{-1} X_n)^{-1}) \to \int fd(Q \circ (\tan^{-1} X)^{-1})$.

Problem 53

Let A and B be any two proper subsets of $\mathbb{R}$. Show that $\mathbb{R}^2 \backslash (A \times B)$ is connected.

Suppose $\mathbb{R}^2 \backslash (A \times B) = (A^c \times \mathbb{R}) \cup (\mathbb{R} \times B^c)$ and any two points if this set can be joined by at most three line segments.

Cor: $(\mathbb{Q} \times \mathbb{Q})^c$ is connected in $\mathbb{R}^2$.

Remark: $\mathbb{R}^2$ can be replaced by the product of arbitrary connected spaces.
Remark: let S be any countable subset of $\mathbb{R}^2$. Let $A = \{a \in \mathbb{R} : (a, b) \in S$ for some $b \in \mathbb{R}\}, B = \{b \in \mathbb{R} : (a, b) \in S$ for some $a \in \mathbb{R}\}$. Then A and B are countable subsets of $\mathbb{R}$ and hence they are proper subsets of $\mathbb{R}$. Thus $\mathbb{R}^2 \backslash (A \times B)$ is connected. Now note that $\mathbb{R}^2 \backslash (A \times B) \subset \mathbb{R}^2 \backslash S$. Also $\mathbb{R}^2 \backslash (A \times B)$ is dense in $\mathbb{R}^2$ because no open ball can be contained in the countable set $A \times B$. It follows that $\mathbb{R}^2 \backslash S$ is connected.
Thus, the complement of any countable set in $\mathbb{R}^2$ is connected.

Problem 54

Find all maps $f : \mathbb{R} \to \mathbb{R}$ such that f is both additive and multiplicative.

We have $f(1) = [f(1)]^2$ so $f(1) = 0$ or $f(1) = 1$. In the second case additivity gives $f(\frac{m}{n}) = \frac{m}{n}$ $\forall n \geq 1$ $\forall m \in \mathbb{Z}$. Now $f(x) = [f(\sqrt{x})]^2 \geq 0$ $\forall x \geq 0$ and $f(x + y) = f(x) + f(y) \geq f(y)$ if $x \geq 0$. Thus f is increasing on $[0, \infty)$. If it is constant on some open interval it is easily seen to be a constant (which must be 1) everwhere. Otherwise it is strictly increasing. We claim that $f(x) = x$ $\forall x$. Let $x > 0$ and $r, s \in \mathbb{Q} \cap (0, \infty)$ with $r < x < s$. Then $r = f(r) < f(x) < f(s) = s$. Letting $r \uparrow x$ and $s \downarrow x$ we get $f(x) = x$. Of course, $f(-x) = -f(x)$ so $f(x) = x$ $\forall x \in \mathbb{R}$. Now let $f(1) = 0$. In this case we get $f(r) = 0$ for all rational r. But f is strictly increasing unless it is a constant. Thus $f(x) = 0$ $\forall x$. Conclusion: $f(x) = x$ $\forall x \in \mathbb{R}$ or $f(x) = 0$ $\forall x \in \mathbb{R}$.

Problem 55

What happens if $\mathbb{R}$ is replaced by $\mathbb{C}$ in Problem 54 and f is assumed to be continuous?

We have $f(rz) = rf(z)$ for all rational r. Again, $f(1) = 0$ or 1. If $f(1) = 0$ then $f(r) = 0$ for all r rational, hence for all r real. Note that $[f(i)]^2 = f(-1) =$

$-f(1) = 0$ so $f(i) = 0$. We now get $f(a+ib) = f(a) + f(i)f(b) = 0$. Thus $f \equiv 0$ in this case. Let $f(1) = 1$. Then $[f(i)]^2 = f(-1) = -f(1) = -1$ so $f(i) = i$ or $f(i) = -i$. In the first case $f(a+ib) = f(a) + f(i)f(b) = a + ib$ and in the second case $f(a+ib) = f(a)+f(i)f(b) = a-ib$. Conclusion: $f \equiv 0$ or $f(z) = z$ $\forall z$ or $f(z) = \overline{z}$ $\forall z$.

Remark: continuity is essential. There exist additive, multiplicative, one-to-one dis-continuous functions on $\mathbb{C}$!.

Remark: if $f : \mathbb{R} \to \mathbb{R}$ is additive and non-measurable then $g = e^{if} : \mathbb{R} \to S^1$ satisfies $g(x+y) = g(x)g(y)$ and g is not measurable. Indeed, if A is a Borel set in S^1 then $B \equiv \{c \in \mathbb{C} : e^{ic} \in A\}$ is Borel in $\mathbb{C}$ and $g^{-1}(A) = f^{-1}(B)$.

Problem 56

Let T be a compact operator on a Hilbert space H with orthonormal basis $\{e_1, e_2, ...\}$. Show that $\|Te_n\| \to 0$.

$\{Te_n\}$ is relatively compact. Any limit point y of this sequence satisfies the property $< y, e_j >= \lim_{l\to\infty} < Te_{n_l}, e_j >= \lim_{l\to\infty} < e_{n_l}, T^*e_j >= 0$ $\forall j$ for some $n_j \to \infty$.

Problem 57

Show that there is a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ converging pointwise which does not converge uniformly on any open interval in $\mathbb{R}$. Show that if a sequence of analytic functions on a region Ω in $\mathbb{C}$ converges pointwise then there is a non-empty open subset D of Ω such that the a subsequence converges uniformly on compact subsets of D.

Let $f(\frac{p}{q}) = \frac{1}{q}$ if $p, q \in \mathbb{Z}, q \geq 1$ and $(p, q) = 1, f(x) = 0$ if x is irrational. We claim that f is upper semi-continuous. If $\alpha > 0$ then $\{x : f(x) < \alpha\}$ is the complement of the (discrete) set of rationals $\frac{p}{q}$ with $p, q \in \mathbb{Z}, 1 \leq q \leq \frac{1}{\alpha}$ and $(p, q) = 1$ and hence it is open. This implies that there is a sequence of continuous functions converging pointwise to f : $\sup\{f(y) - n|x-y|\}, n = 1, 2, ...$ is one such sequence. Since f is not continuous at rationals the sequence cannot converge uniformly to f on any open interval.

If $f_n(z) \to f(z)$ $\forall z \in \Omega$ where each f_n is analytic on Ω then $\Omega = \bigcup_N \{z \in \Omega : |f_n(z)| \leq N\}$ and Baire Category Theorem implies that $\{f_n\}$ is uniformly bounded on some open ball B contained in Ω. The sequence $\{f_n\}$ is normal in B and hence it has a subsequence converging uniformly on compact subsets of B.

Problem 58

Let μ be a finite positive measure on $(1,\infty)$ and $f(y) = \int\limits_1^\infty \cos(xy)d\mu(x)$. Show that f has at least one zero on $[0,\pi]$.

Consider $\int\limits_0^\pi \int\limits_1^\infty \cos(xy)d\mu(x)\sin y dy$. By Fubini's Theorem this is $\int\limits_1^\infty \int\limits_0^\pi \sin y \cos(xy)dyd\mu(x) =$ $\frac{1}{2}\int\limits_1^\infty \int\limits_0^\pi [\sin(y(1+x))+\sin(y(1-x))]dyd\mu(x) = \frac{1}{2}\int\limits_1^\infty [\frac{1-\cos\pi(1+x)}{1+x}+\frac{1-\cos\pi(1-x)}{1-x}]d\mu(x)$ $= \frac{1}{2}\int\limits_1^\infty [\frac{1+\cos\pi x}{1+x} + \frac{1+\cos\pi x}{1-x}]d\mu(x) = \int\limits_1^\infty \frac{1+\cos\pi x}{1-x^2}d\mu(x) < 0$. It follows that $\int\limits_0^\pi f(y)\sin y dy < 0$. If f has no zero in $[0,\pi]$ then it does not change sign and since $f(0) > 0$ it has to be positive throughout the interval which forces $\int\limits_0^\pi f(y)\sin y dy$ to be positive.

Problem 59

Consider the following sets of 3×3 real matrices:
a) $\{A : \det(A) = 0\}$
b) $\{A : A \text{ is symmetric}\}$
c) $\{A : A^n = 0 \text{ for some } n \in \mathbb{N}\}$
Treating a 3×3 real matrix as an element of $\mathbb{R}^9$ show that above sets of Lebesgue measure 0.

The set in b) is a proper linear subspace of $\mathbb{R}^9$ and hence it has measure 0. The set in c) is comtained in the one in a). To show that the set in a) has measure 0 expand the determinant using the first row and use Fubuni's theorem. If the 2×2 obtained by deleting the first row and first column is non-zero then, for fixed values of a_{ij} with $(i,j) \neq (1,1)$ there is only one point in our set and so it has measure 0. The matrices A with $\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = 0$ have measure 0 (again by Fubini's Theorem) unless all the entries are 0. As longas A is not the zero amtrix there is a 2×2 sub-matrix for which Fubuni's Theorem can be applied.

Problem 60

Let $\{f_n\}$ be an orthonormal set in $L^2([0,1])$ and $A = \{x : \lim\limits_{n\to\infty} f_n(x) \text{ exists}\}$ and let $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for $x \in A$. Show that $f = 0$ a.e. on A.

Let $\epsilon > 0$. There is a set $A_k \subset A$ such that $f_n \to f$ uniformly on A_k and $m(A \backslash A_k) < \frac{1}{k}$. Now $\int I_E I_{A_k} f_n \to 0$ as $n \to \infty$ for each k and each measurable set E. This implies $\int I_E I_{A_k} f = 0$ for each k. [Note that $\int\limits_A |f| \leq \liminf\limits_n \int\limits_A |f_n| \leq \liminf\limits_n (\int |f_n|^2)^{1/2} \sqrt{m(A)} \leq 1$ so f is integrable on A]. It now folloes that $\int I_E I_A f = 0$ for each measurable set E, so $f = 0$ a.e.

Problem 61

Let $T : l^\infty \to \mathbb{R}$ be a linear map such that for any $x = \{x_n\} \in l^\infty$, $T(x) = \lim x_{n_j}$ for some subsequence $\{n_j\}$ of $\{1, 2, ...\}$. Show that T is continuous and multiplicative.

Say $x \leq y$ if $x_n \leq y_n$ for each n. Write 0 for $\{0, 0, ...\}$, 1 for $\{1, 1, ...\}$, xy for $\{x_n y_n\}$. We have $T(1) = 1$ and $T(x) \geq 0$ if $x \geq 0$ so $-\|x\|_\infty \leq T(x) \leq \|x\|_\infty$ and T is continuous. We can approximate any $x \in l^\infty$ be a sequence whose components take only finite number of values. [Simple function approximation of bounded measurable functions]. Any x whose components take only finite number of values is a linear combination of sequence whose components take only the values 0 and 1. Hence it suffices to show that $T(xy) = T(x)T(y)$ when x and y are $0-1$ sequences. If $Tx = 0$ or $Ty = 0$ then $T(xy) = 0$ because $xy \leq x$ and $xy \leq y$. Suppose $Tx = 1$ and $Ty = 1$. Then $(x-y)^2$ is also a $0-1$ valued sequence and $T(x-y)^2 = Tx^2 + Ty^2 - 2T(xy) = Tx + Ty - 2T(xy) = 2$ if $T(xy) = 0$. This is a contradiction and hence $T(xy) = 1$.

Problem 62

Let $c_1, c_2, ..., c_n$ be distinct complex numbers. Show that $\sum\limits_{k=1}^{n} \prod\limits_{j \neq k} \frac{c_j - c}{c_j - c_k} = 1$ for all $c \in \mathbb{C}$.

The left side is a polynomial of degree $(n-1)$ which has the vale 1 at each of the points $c_1, c_2, ..., c_n$.

Problem 63

Compute $\limsup |a^n - b^n|^{1/n}$ for any two complex numbers a and b.

The radius of convergence of $\sum\limits_{n=0}^{\infty} (a^n - b^n) z^n$ is the maximum of the radii of convergence of $\sum\limits_{n=0}^{\infty} a^n z^n$ and $\sum\limits_{n=0}^{\infty} b^n z^n$ and hence the answer is $\max\{|a|, |b|\}$.

Problem 64

Prove the identity $[x]+[x+1/n]+...+[x+\frac{n-1}{n}]=[nx]$ for all $x\in\mathbb{R}, n\in\mathbb{N}$.

Let $f(x)=[x]+[x+1/n]+...+[x+\frac{n-1}{n}]-[nx]$. Note that $f(x+1/n)=f(x)$ $\forall x$. On $[0,1/n)$ we have $f(x)=0+0+...+0-0=0$.

Problem 65

If $f:[0,1]\to\mathbb{R}$ satisfies $|f(x)-f(y)|\leq C\,|x-y|$ $\forall x,y$ prove that given $\epsilon>0$ there is a polynomial p such that $|p(x)-p(y)|\leq C\,|x-y|$ $\forall x,y$ and $|f(x)-p(x)|<\epsilon$ $\forall x$.

f is absolutely continuous, so f' exists a.e. and f' is integrable. Further $|f'(x)|\leq C$ a.e.. There is a continuous function g such that $|g(x)|\leq c$ $\forall x$ and $\int |f'(x)-g(x)|<\epsilon$. There is a polynomial q such that $|q(x)-g(x)|<\epsilon$ $\forall x$ and $|q(x)|\leq C$ $\forall x$. Let $p(x)=f(0)+\int\limits_0^x q(t)dt$. Then p is a polynomial, $|p(x)-p(y)|\leq C\,|x-y|$ $\forall x$ and $|f(x)-p(x)|=\left|\{f(0)+\int\limits_0^x f'(t)dt\}-\{f(0)+\int\limits_0^x q(t)dt\}\right|\leq$ $\int\limits_0^1 |f'(t)-q(t)|\,dt<\epsilon+\int\limits_0^1 |f'(t)-g(t)|\,dt<2\epsilon$ $\forall x$.

Problem 66

If $f:[0,\pi]\to\mathbb{R}$ is continuous and $\int\limits_0^\pi f(x)\sin x dx=\int\limits_0^\pi f(x)\cos x dx=0$ show that f has at least two zeros in $[0,\pi]$.

Since $\sin x\geq 0$ on $[0,\pi]$ we may suppose that f takes both positive and negative values and hence has at least one zero. Suppose it has only one zero a. Then $\int\limits_0^\pi f(x)\sin(x-a)dx=0$. Hence $\int\limits_0^a f(x)\sin(x-a)dx+\int\limits_a^\pi f(x)\sin(x-a)dx=0$. Since $\sin(x-a)<0$ on $[0,a]$ and >0 on $[a,\pi]$ and f does not change sign in either of these intervals it has to have the same constant sign in these two intervals for above equation to hold. this is a contradiction since f takes both positive and negative values.

Problem 67

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and non-increasing show that it has a unique fixed point. Use this to show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = -x \ \forall x \in \mathbb{R}$.

Let f be non-increasing. If $x < y$ are fixed points then $x = f(x) \geq f(y) = y > x$, a contradiction. If there are no fixed points then either $f(x) > x \ \forall x$ or $f(x) < x \ \forall x$. In the first case $f(x) \to \infty$ as $x \to \infty$. However $x > 1$ implies $1 < x < f(x) \leq f(1)$ which leads to a contradiction by letting $x \to \infty$. Similarly in the second case $f(x) \to -\infty$ as $x \to -\infty$ and $x < 1$ implies $f(1) \leq f(x)$, a contradiction. Hence f has a unique fixed point.

Now let f be a continuous function $: \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = -x \ \forall x \in \mathbb{R}$. We first observe that f is one-to-one. Indeed, $f(x) = f(y) \Rightarrow -x = f(f(x)) = f(f(y)) = -y$. Thus f is strictly monotonic on $\mathbb{R}$. If it is strictly increasing then so is $f \circ f$ but this contradicts the fact that $f(f(x)) = -x \ \forall x$. Hence f is strictly decreasing and the first part shows that it has a unique fixed point a. But $-a = f(f(a)) = a$ so $a = 0$. It follows that $f(x) - x$ does not change sign in $(0, \infty)$ as well as in $(-\infty, 0)$. Since $f(0) = 0$ and f is strictly decreasing we see that $f(x) < 0$ for $x > 0$ and $f(x) > 0$ for $x < 0$. Now $f(1) < 0$ so $f(f(1)) > 0$ which leads to the contradiction $-1 > 0$.

Remarks: for any n the only continuous function f on $\mathbb{R}$ whose $n-th$ iterate $f_{(n)}$ is the identity function is the identity function itself. The only continuous function f on $\mathbb{R}$ such that $f_{(n)}(x) = -x \ \forall x$ is $-x$ if n is odd and there is no such function if n is even.

Problem 68

If $f : [0,1] \to \mathbb{R}$ is continuous show that $\frac{1}{n} \sum\limits_{j=1}^{n} (-1)^j f(\frac{j}{n}) \to 0$ as $n \to \infty$.

This follows by writing the sum in terms of $f(\frac{j-1}{n}) - f(\frac{j}{n})$ and using uniform continuity.

Problem 69

If n is a positive integer find the precise number of real roots of the equation $\sum\limits_{k=0}^{n} \frac{x^k}{k!}$.

If n is even then $e^{-x} < \sum\limits_{k=0}^{n} \frac{(-x)^k}{k!}$. This shows that the given polynomial has no roots in $(-\infty, 0)$ in this case. Of course, $\sum\limits_{k=0}^{n} \frac{x^k}{k!} \geq 1$ if $x \geq 0$ so there are no real roots for n even. Now let n be odd. Since $\sum_{k=0}^{n} \frac{x^k}{k!} \to \infty$ as $x \to \infty$ and $\sum_{k=0}^{n} \frac{x^k}{k!} \to -\infty$ as $x \to -\infty$ it follows that $\sum_{k=0}^{n} \frac{x^k}{k!} = 0$ for some x. If there are two real roots then the derivative of $\sum_{k=0}^{n} \frac{x^k}{k!}$ must vanish at some point, but that is a contradiction to the fact that the given polynomial has no real roots for n even.

Problem 70 (universal power series)

Show that there is a power series $\sum_{k=1}^{\infty} c_n x^n$ (with no constant term) such that for any continuous function $f : [0,1] \to \mathbb{C}$ with $f(0) = 0$ there is a subsequence $\{s_{n_k}\}$ of the sequence of partial sums of this series converging uniformly to f on $[0,1]$.

Remark: an arbitrary continuous function cannot expressed in the form $\sum_{k=0}^{\infty} c_n x^n$ with the series converging pointwise. Such a representation would force f to be the restriction to $[0,1)$ of an analytic function on $\{z : |z| < 1\}$.

Let X be the space of all continuous functions $f : [0,1] \to \mathbb{C}$ with $f(0) = 0$. Give X the supremum metric. Then, for any positive integer k polynomials of the type $\sum_{m=1}^{n} a_m x^{km}$ (where $n \geq 1, a_k' s \in \mathbb{C}$) are dense in X. (It is an easy consequence of Stone-Weierrstrass Theorem that polynomials of the type $\sum_{m=0}^{n} a_m x^{km}$ are dense in $C[0,1]$. If $f \in C[0,1]$ and $f(0) = 0$ then we can omit the constant term). Now let $\{f_n\}$ be a countable dense subset of X. Let p_1 be a polynomial without cosntant term such that $\|f_1 - p_1\| < \frac{1}{2}$. ($\|\|$ is the supremum norm). Let $d_1 = \deg(p_1)$ and d_2 be an integer $> 1 + d_1$. Let p_2 be a polynomial without cosntant term such that $\left|f_2(x) - p_1(x) - p_2(x^{d_2})\right| < \frac{1}{2^2}$ $\forall x$. By induction we get polynomials $p_1, p_2, ...$ and an increasing sequence of integers $d_1, d_2, , , ,$ such that $\left|f_n(x) - p_1(x) - p_2(x^{d_2}) - ... - p_n(x^{d_k})\right| < \frac{1}{2^n}$ $\forall x \forall n$. The required power series is $p_1(x) + p_2(x^{d_2}) + p_3(x^{d_3}) + ...$

Problem 71

Show that $\int_{-\infty}^{\infty} (\frac{\sin x}{x})^{2n} \cos(2xy) dx = 0$ if $|y| > 2n$. Also show that the integral is > 0 for all other values of y.

Note that $\int_{-\infty}^{\infty} I_{[-1,1]}(t)e^{itx}dt = 2\frac{\sin x}{x}$ for $x \neq 0$. (The equation holds for $x = 0$ also if we interpret $\frac{\sin x}{x}$ as 1 when $x = 0$). Let f be the $n-$fold convolution of $I_{[-1,1]}$. Then $\int_{-\infty}^{\infty} f(t)e^{itx}dt = 2^n(\frac{\sin x}{x})^n$. By the inversion formula $\int_{-\infty}^{\infty} (\frac{\sin t}{t})^n e^{itx}dt = (2\pi)2^{-n}f(x)$. Clearly, $f(x) > 0$ if $|x| < n$ and $f(x) = 0$ if $|x| \geq n$.

Problem 72

Let $f \in C[0,1]$ and $f(0) = 0$. Show that there is a sequence of polynomials $p_n(x) = \sum_{k=1}^{k_n} a_{k,n}x^k$ converging pointwise to f on $[0,1]$, uniformly on $[\delta, 1]$ $\forall \delta \in (0,1)$, such that $a_{k,n} \to 0$ as $n \to \infty$ for every $k \in \mathbb{N}$.

Let $|f(x)| < \frac{1}{n}$ for $0 \leq x \leq t_n$. Let $q_n(x) = \sum_{k=1}^{k_n} b_{k,n}x^k$ be such that $\left|\frac{f(x)}{x^n} - \sum_{k=0}^{k_n} b_{k,n}x^k\right| < \frac{1}{n}$ for $t_n \leq x \leq 1$. Let $p_n(x) = x^n q_n(x)$.

Problem 73

If $f : (0,1) \to (0,\infty)$ is decreasing show that $\frac{\int_0^1 xf^2(x)dx}{\int_0^1 xf(x)dx} \leq \frac{\int_0^1 f^2(x)dx}{\int_0^1 f(x)dx}$.

We have $\int_0^x (x-y)f^2(y)dy \geq f(x)\int_0^x (x-y)f(y)dy$. This shows that the derivative of the function $(\int_0^x yf^2(y)dy)(\int_0^x f(y)dy) - (\int_0^x f^2(y)dy)(\int_0^x yf(y)dy)$ is ≤ 0. Hence the value of this function at $x = 1$ does not exceed its value at 0.

Problem 74

If f and g are continuous functions on $(0,1)$ and $g(x) > 0$ $\forall x$ show that $\lim\limits_{n\to\infty} \dfrac{\int\limits_0^1 x^n f(x)dx}{\int\limits_0^1 x^n g(x)dx}$ exists.

We have $\dfrac{\int\limits_0^1 x^n f(x)dx}{\int\limits_0^1 x^n g(x)dx} = \dfrac{\int\limits_0^{1-\delta} x^n f(x)dx + \int\limits_{1-\delta}^1 x^n f(x)dx}{\int\limits_0^{1-\delta} x^n g(x)dx + \int\limits_{1-\delta}^1 x^n g(x)dx} \leq \dfrac{(1-\delta)^n\alpha + (f(1)+\epsilon)\frac{1}{n+1}}{(g(1)-\epsilon)\int\limits_{1-\delta}^1 x^n dx - (1-\delta)^n\beta}$

for $\delta = \delta(\epsilon)$ sufficiently small, where $\alpha = \int |f(x)|\, dx$ and $\beta == \int |g(x)|\, dx$. Using the fact that $(n+1)(1-\delta)^n \to 0$ as $n \to \infty$ we conclude that $\limsup \dfrac{\int\limits_0^1 x^n f(x)dx}{\int\limits_0^1 x^n g(x)dx} \leq \frac{f(1)}{g(1)}$. A similar argument shows that $\liminf \dfrac{\int\limits_0^1 x^n f(x)dx}{\int\limits_0^1 x^n g(x)dx} \geq \frac{f(1)}{g(1)}$.

Problem 75

Say that two functions $f, g : \mathbb{R} \to \mathbb{R}$ are similar if there is a bijection $\phi : \mathbb{R} \to \mathbb{R}$ such that $f = \phi^{-1} \circ g \circ \phi$. Prove that x^n and x^m are similar if $n = m^k$ for some k (or $n = m^k$ for some k). Are x^2 and x^2+1 similar? Prove that x^n and x^m are similar if n and m are both odd and greater then 1. Prove that sin and cos are not similar.

First part: define $\phi(x)$ to be $e^{(\log(x)^j}$ if $x > 0, -e^{(\log(x)^j}$ if $x < 0, 0$ if $x = 0$. Second part is easy: if there is a bijection ϕ such that $\phi(x^2) = [\phi(x)]^2 + 1$ then $\phi(t) \geq 1$ $\forall t \geq 0$. But $[\phi(-x)]^2 = \phi^2(x)$ so $\phi(-x) = \pm\phi(x)$. It follows that $|\phi(t)| \geq 1$ for every real number t which implies that ϕ is not onto $\mathbb{R}$. Thus x^2 and x^2+1 are not similar. For the third part we claim that there is a bijection h on $\mathbb{R}$ such that $h(x + \log n) \equiv \log m + h(x)$. For this we take any bijection $h : (0, \log n] \to (0, \log m]$ and define $h(x + (\log n)j) = h(x) + (\log m)j$ for $j = 1, 2, ...$ to get a bijection h of $(0, \infty)$. Defining $h(0)$ to be 0 and defining

h on $(-\infty, 0)$ in a similar fashion we get the desired bijection h. We now define $\phi(x) = e^{h(\log x)}(x > 0)$, $-e^{h(\log x)}(x < 0)$ and $\phi(0) = 0$. Then ϕ is a bijection of $\mathbb{R}$ and $\phi(nx) = m\phi(x)\ \forall x$. Finally, let $f(x) = e^{\phi(\log x)}(x > 0)$, $-e^{\phi(\log x)}(x < 0)$ and $f(0) = 0$. Then f is also a bijection and $f(x^n) = [f(x)]^m$ which proves that x^n and x^m are similar. We now prove that sin and cos are not similar. Suppose ϕ is a bijection such that $\phi(\cos(x)) = \sin(\phi(x))$. Then $\phi([-1,1]) = [-1,1]$. Since $\cos x = \phi^{-1}(\sin(\phi(x)))$ and the right side is $1-1$ on $[-1,1]$ it follows that cos is $1-1$ on $[-1,1]$ which contradicts the fact that *cos* is even.

Problem 76

Show that there is a sequence of polynomials converging pointwise, but not uniformly, to a *continuous* function on $[0,1]$.

$x^n - x^{n^2}$. Note that $(1-\frac{1}{n})^n - (1-\frac{1}{n})^{n^2} \to e$. [For the same question on $\mathbb{R}$ we can take $p_n(x) = \frac{x}{n}$].

Problem 77

a) Prove or disprove: if $f : \mathbb{R} \to \mathbb{R}$ is a function such that $\{(x,y) : y \neq f(x)\}$ is open then f is continuous.

b) Prove or disprove: if $f : \mathbb{R} \to \mathbb{R}$ is a function such that $\{(x,y) : y > f(x)\}$ and $\{(x,y) : y < f(x)\}$are open then f is continuous.

First statement is false and $f(x) = \frac{1}{x}, x \neq 0, f(0) = 0$ is a counterexample. For the second part we consider $\{(x,y) : y > f(x)\} \cap \{(x,y) : y < a)\}$ and project this to $\mathbb{R}$ to see that $\{x : f(x) < a\}$ is open for each $a \in \mathbb{R}$. Similarly, the projection of $\{(x,y) : y < f(x)\} \cap \{(x,y) : y > a)\}$, which is $\{x : f(x) > a\}$, must be open. It follows that $f^{-1}(U)$ is open for every open interval U, hence for every open set U.

Remark: the answer to a) changes if we assume that f is bounded. It also changes if the graph is assumed to be connected. [cf. Gelbaum, Problems in Analysis. See also problem 101 below].

Problem 78

Let (X,d) be a metric space. Show that X is separable if and only if there is an equivalent metric on it which makes it totally bounded.

Let X be separable. Let $\{x_n\}$ be a countable dense set. The map $f : X \to [0,1]^{\mathbb{N}}$ defined by $f(x) = (\frac{d(x,x_1)}{1+d(x,x_1)}, \frac{d(x,x_2)}{1+d(x,x_2)}, ...)$ is a homeomorphism of X into $[0,1]^{\mathbb{N}}$. Define a new metric D on X by $D(x,y) = d_0(f(x), f(y))$ where the metric d_0 on $[0,1]^{\mathbb{N}}$ is defined by $d_0(\{a_n\},\{b_n\}) = \sum\limits_{k=1}^{\infty} \frac{|a_n-b_n|}{1+|a_n-b_n|}\frac{1}{2^n}$. Since the range of f is relatively compact, it is totally bounded. Hence, (X, D) is totally bounded. D is equivalent to d because f is a homeomorphism.

The converse part is fairly straightforward: cover X by a finite number of balls of radius n for $n = 1, 2, ...$ and verify that the centers of these balls form a countable dense set.

Remark: it is clear from above proof that the two equivalent conditions are also equivalent to the existence of a compact metric space Y such that X is homeomorphic to a subset of Y.[In other words, X has a metrizable compactification].

Problem 79

Let $f : \mathbb{R} \to \mathbb{R}$ be additive. Show that the following statements are equivalent:

a) f is continuous

b) $f^{-1}\{0\}$ is closed

c) f is bounded on some open interval containing 0

d) $f(U)$ is not dense in $\mathbb{R}$ for some open set U containing 0

e) f is Lebesgue measurable

a) implies b) is obvious. Let b) hold and suppose c) does not hold. Then there is a sequence $\{t_n\} \to 0$ such that $|f(t_n)| \to \infty$. Fix $y \in \mathbb{R}$ with $f(y) \neq 0$ and consider the numbers $y - \frac{f(y)}{f(t_n)} t_n$. Since this sequence converges to y we get $f(y) = 0$ (by b)) which is a contradiction. c) implies d) is obvious. Now we prove (the interesting part) that d) implies c). If c) is false there is a sequence $\{t_n\} \to 0$ such that $|f(t_n)| \to \infty$. Let $s_n = f(t_n)$. Since $f(-x) = -f(x)$ we may suppose $s_n \to +\infty$. If $x > 0$ and $\epsilon > 0$ is sufficiently small then, for n sufficiently large, the length $\frac{s_n}{x-\epsilon} - \frac{s_n}{x+\epsilon}$ of the interval $(\frac{s_n}{x+\epsilon}, \frac{s_n}{x-\epsilon})$ exceeds 1 and hence it contains an integer k. Thus, $s_n < k(x + \epsilon)$ and $k(x - \epsilon) < s_n$. This gives $\left|\frac{f(t_n)}{k} - x\right| < \epsilon$. Hence, the interval $(x - \epsilon, x + \epsilon)$ contains $f(\frac{t_n}{k})$ which belongs to $f(-\delta, \delta)$ if n is sufficiently large. We have proved that $f(-\delta, \delta)$ intersects every open interval contained in $(0, \infty)$ and hence it is dense in $(0, \infty)$. The fact that $f(-x) = -f(x)$ now shows that the image of every interval around 0 is dense in $\mathbb{R}$. We have now proved $a) \Rightarrow b) \Rightarrow c) \Rightarrow d) \Rightarrow c)$. $c) \Rightarrow a)$ is elementary: if $|f(x)| \leq M$ for $|x| \leq \delta$ then $|f(y) - f(x)| < \epsilon$ if $|y - x| < \delta/k$ and k is so large that $\frac{M}{k} < \epsilon$. Finally we prove e) implies c). [Of course, c) implies a) and a) implies e)]. If N is sufficiently large then $E = \{x : |f(x)| < N\}$ has positive Lebesgue measure. Hence, there exists $\delta > 0$ such that $(-\delta, \delta) \subseteq \{x - y : |f(x)| < N$ and

$|f(y)| < N\}$. It follows, by additivity, that $(-\delta, \delta) \subseteq \{x : |f(x)| < 2N\}$. Hence c) holds.

Problem 80

Let (X, d) be a metric space. Consider the following properties of X :
a) Every real continuous function on X is bounded
b) Every real continuous bounded function on X attains its supremum
c) Every real continuous function on X is uniformly continuous
d) The image of every real continuous function on X is connected
e) $d(A, B) > 0$ whenever A and B are disjoint closed sets in X
Do any of the first the conditions a),b),c),e) imply that X is compact? Does d) imply that X is connected?

The answers are all YES except for c) and e). $\mathbb{N}$ is a counter-example for c) and e). b) requires a form of

's Theorem where the range is an open interval in $\mathbb{R}$.See problem 217 below.

Problem 81

a) Suppose $f : \mathbb{R} \to \mathbb{R}$ has a left limit $f(x-)$ at every point and suppose $f(x-)$ is continuous at a. Does it follow that f is continuous at a? What if $f(x-) \to f(a)$ as $x \to a$?

b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and has a left derivative $f'(x-)$ at every point. Suppose $f'(x-)$ is continuous at a. Show that f is differentiable at a.

If $f(x) = x \; \forall x \neq 0$ and $f(0) = 1$ then $f(x-) = x \; \forall x$ and f is not continuous. Suppose $f(x-) \to f(a)$ as $x \to a$. Then given $\epsilon > 0$ there is a $\delta > 0$ such that $f(a) - \epsilon \leq f(x-) \leq f(a) + \epsilon$ for $a - \delta \leq x \leq a + \delta$. We claim that $f(a) - \epsilon \leq f(x) \leq f(a) + \epsilon$ for $a - \delta < x < a + \delta$. This would complete the proof. Suppose $f(x_0) < y < f(a) - \epsilon$ with $a - \delta < x_0 < a + \delta$. Consider $u = \inf\{x \in (a - \delta, a + \delta) : f(x) > y\}$. Since $f(x) \leq y$ for $x < u$ we have $f(u-) \leq y < f(a) - \epsilon$ which is a contradiction. This proves that x_0 does not exist which means $f(a) - \epsilon \leq f(x)$ for $a - \delta < x < a + \delta$. Similarly we get $f(x) \leq f(a) + \epsilon$ for $a - \delta < x < a + \delta$.

For the proof of b) we proceed in a similar way: let $f'(a-) - \epsilon \leq f'(x-) \leq f'(a-) + \epsilon$ for $a - \delta \leq x \leq a + \delta$ and $f(x_0 - h_0) - f(x_0) < yh_0 < (f(a) - \epsilon)h$ we consider $h_1 = \inf\{h : f(x_0 - h) - f(x_0) \geq yh\}$ to get $f'((x_0 + h_1)-) < (f(a) - \epsilon)h$, a contradiction.

Problem 82

Let $f : \mathbb{R} \to \mathbb{R}$ be a function which has a local minimum at each point. Show that its range is atmost countable. Construct an example of such a function which is increasing and which has the properties $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = -\infty$. If f has a local minimum at each point and if f is also continuous show that it is a constant.

Let $A_n = \{x : f(y) \geq f(x)\ \forall y \in (x - \frac{1}{n}, x + \frac{1}{n})\}$. Then $f(\mathbb{R}) = \bigcup_n f(A_n)$ and each $f(A_n)$ is at most countable: note that $f(A_n) = \bigcup_k f(A_n \cap [-k, k])$ and $[-k, k]$ can be covered by a finite number of intervals of length $\frac{1}{2n}$; if $x, y \in A_n$ and $|x - y| < \frac{1}{n}$ then $f(x) \geq f(y)$ and $f(y) \geq f(x)$, so $f(x) = f(y)$. Thus f is constant in each of these sub-intervals, so $f(A_n \cap [-k, k])$ is finite. This proves the first part. If $f(x) = a_n$ for $-n < x \leq n + 1$ with $a_n \leq a_{n+1} (n \in \mathbb{Z})$ then f has a local minimum at each point. This answers the second part. Suppose f has a local minimum at each point and if f is also continuous. Since the range of f is a countable connected set it must be a singleton.

Problem 83

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = f(x)\ \forall x$. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = f(x)\ \forall x$. If f is a non-constant convex function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = f(x)\ \forall x$ show that it is identity on $[a, \infty)$ for some real number a and give an example of such a function. Prove that there is no differentiable function $f : \mathbb{R} \to \mathbb{R}$ other then the identity such that $f(f(x)) = f(x)\ \forall x$.

Let $A \subset \mathbb{R}, f : A^c \to A$ any function and $f(x) = x\ \forall x \in A$. Then $f(f(x)) = f(x)\ \forall x$. Given any $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = f(x)\ \forall x$ take A to be $f(\mathbb{R})$. This solves the first part. If f is continuous then A is an interval and $f(x) = x\ \forall x \in \bar{A}$. $f : (\bar{A})^c \to A$ can be arbitrary. This answers the second part. If f is convex the A is an interval of positive length. If A is bounded, say, with end points a and b then, for $x > b$ we have $f(b) \leq \lambda f(a) + (1 - \lambda) f(x)$ where λ is defined by $b = \lambda a + (1 - \lambda)x$. Thus, $f(x) \geq \frac{f(b) - \lambda f(a)}{1 - \lambda} = \frac{b - \lambda a}{1 - \lambda} > b$ (where we have used the fact that f is identity on $[a, b]$). Thus f does not take values in $[a, b]$ and we have the desired contradiction. A similar argument shows that A cannot be bounded above. Thus f is identity on $[a, \infty)$ for some real number a. Examples of such function are x^+ and $|x|$. Finally if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies $f(f(x)) = f(x)\ \forall x$ the the range A has to be $\mathbb{R}$. To see this note that A is an interval and if it has finite supremum b then $f'(b) = 1$ which forces f to take values exceeding b at points close to b and greater than b. This contradicts the fact that f takes values in $[a, b]$. A similar argument shows that A cannot be bounded below either. Thus $A = \mathbb{R}$ and $f(x) = x\ \forall x$.

Problem 84

Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and $f : X \to Y$. Prove or disprove the following:

a) if $(f^{-1}(A))^0 \neq \emptyset$ whenever $A^0 \neq \emptyset$ then f is continuous

b) if $X_1 = X_2 = X$ (say) and a set A is sense in X w.r.t. τ_1 if and only if it is dense in X w.r.t. τ_2 then $\tau_1 = \tau_2$.

c) if $(f(A))^0 \neq \emptyset$ whenever $A^0 \neq \emptyset$ then f is an open map

All the three statements are false. Let τ_1 be the usual topology on $\mathbb{R}$ and τ_2 be the class of all possible unions of intervals of one of the following types: $[0,1], [0,a)$ with $0 < a \leq 1, (b,1]$ with $0 \leq b < 1, (a,b)$ with $-\infty < a < b < \infty$.

Note that τ_2 is the smallest topology containing all the open sets in the usual tolpology and the interval $[0,1]$.

Let $f : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$ be the identity map. Then f is not continuous because $f^{-1}([0,1])$ is not open. However, if A has nonempty interior under τ_2 then it contains one of the intervals mentioned above and so it has non-empty interior in τ_1. This completes a). b) is fasle by the same example. For c) we just have to look at the identity map $(\mathbb{R}, \tau_2) \to (\mathbb{R}, \tau_1)$.

Problem 85

Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that the smallest topology that makes f continuous (w.r.t the usual topology on the range) is the power set of $\mathbb{R}$?

Answer: no. If so then for each x there is an open set U_x such that $\{x\} = f^{-1}(U_x)$. Let $U = \bigcup\limits_{x \in \mathbb{R}} U_x$. We can write U as a countable union, say $\bigcup\limits_{n \in \mathbb{N}} U_{x_n}$. If $x \in \mathbb{R}$ then $f(x) \in U$ (because $x \in f^{-1}(U_x) \subset f^{-1}(U)$. Thus, $f(x) \in U_{x_n}$ for some n. Hence $x \in f^{-1}(U_{x_n}) = \{x_n\}$. We have proved that $\mathbb{R} \subset \{x_1, x_2, ...\}$ which is a contradiction.

Problem 86

Prove that a function f from one metric space to another is uniformly continuous if and only if $d(A, B) = 0$ implies $d(f(A), f(B)) = 0$.

Solution by Suresh Nayak when the domain and range are both equal to $\mathbb{R}$:

It is easy to see that f is continuous: if not $\exists x_n \to x$ with $|f(x_n) - f(x)| \geq \delta > 0$ and we get a contradiction by taking $A = \{x_n : n \geq 1\}, B = \{x\}$. If f is not uniformly continuous then we can find $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \delta > 0 \ \forall n$. We may suppose that $x_n < y_n \ \forall n$. Claim: $\{(x_n, y_n) : n \geq 1\}$ is unbounded in $\mathbb{R}^2$. If the claim is false there would be integers $n_1 < n_2 < ...$ such that $\{(x_{n_k}, y_{n_k})\}$ converges to some point (x, y). Since $|x_n - y_n| < \frac{1}{n}$ we get $x = y$. But then $f(x_{n_k}) \to f(x), f(y_{n_k}) \to f(x)$ and $|f(x_{n_k}) - f(y_{n_k})| \geq \delta$ leading to a contradiction. The claim is proved and, by going to a subsequence we may suppose $|x_{n+1}| > |x_n| + 1$. [Note that since $|x_n - y_n| < \frac{1}{n}$ both $\{x_n\}$ and $\{y_n\}$ are unbounded]. Let $A = \{x_n : n \geq 1\}$ and $B = \{y_n : n \geq 1\}$. Suppose $f(B)$ is bounded. we may suppose that $f(y_n) \to \lambda$

(say). Thus the diameter of the set $f(\{y_n : n \geq k\})$ is less than $\delta/2$ if k is large enough. Since the distance between $\{x_n : n \geq k\}$ and $\{y_n : n \geq k\}$ is 0 the distance between $f(\{x_n : n \geq k\})$ and $f(\{y_n : n \geq k\})$ is also 0. However $|f(x_n) - f(y_n)| \geq \delta$ so $|f(x_n) - f(y_j)| \geq \delta - |f(y_j) - f(y_n)| \geq \delta/2$ for all $n, j \geq k$, a contradiction. We have proved that $f(B)$ is unbounded. We may assume now that $f(y_n)$ is monotonic with $|f(y_{n+1}) - f(y_n)| > \delta$. Since f is continuous and $|f(x_n) - f(y_n)| \geq \delta$ we can find $t_n \in [x_n, y_n]$ such that $|f(t_n) - f(y_n)| = \delta/2$. Now $|f(t_n) - f(y_m)| \geq |f(y_n) - f(y_m)| - |f(t_n) - f(y_n)| > \delta - \delta/2 = \delta/2$ whenever $n \neq m$. Thus the distance between $\{f(t_n) : n \geq 1\}$ and $\{f(y_n) : n \geq 1\}$ is positive whereas the distance between $\{t_n : n \geq 1\}$ and $\{y_n : n \geq 1\}$ is 0.

Solution (general case) by Kannappan (student of B. Math III):

if f is not uniformly continuous then there exists $\epsilon > 0$ such that for every $\delta > 0$ we can find points x and y with $d(x, y) < \delta$ but $d(f(x), f(y)) \geq 4\epsilon$. Let $\delta_1 = 1$.

Let $d(x_1, y_1) < \delta$ but $d(f(x_1), f(y_1)) \geq 4\epsilon$. Inductively define $\delta_n, x_n, y_n (n \geq 1)$ satisfying the conditions $\delta_n \downarrow 0, d(x_n, y_n) < \delta_n, d(f(x_n), f(y_n)) \geq 4\epsilon$ as follows: having found $\delta_j, x_j, y_j (j \leq n)$ let $0 < \delta_{n+1} < \min\{\frac{\delta_n}{2}, d(N(x_n), F(x_n)), d(N(y_n), F(y_n))\}$ where $N(z) = \{x : d(f(x), f(z)) < \epsilon\}$ and $F(z) = \{x : d(f(x), f(z)) \geq 2\epsilon\}$ for any z. Note that $z \in N(z)$. If $F(z) = \emptyset$ then $d(f(x), f(z)) < 2\epsilon$ for all x and $4\epsilon \leq d(f(x_1), f(y_1) \leq d(f(x_1), f(z)) + d(f(y_1), z) < 2\epsilon + 2\epsilon$, a contradiction. Hence $F(z)$ and $N(z)$ are both non-empty for any z. Also, $d(N(x_n), F(x_n))$ and $d(N(y_n), F(y_n))$ are both $\geq \epsilon > 0$: if $u \in F(z)$ and $v \in N(z)$ for some z then $d(f(u), f(v)) \geq d(f(u), f(z)) - d(f(v), f(z)) > 2\epsilon - \epsilon = \epsilon$. Hence δ_{n+1} is well defined. We can find x_{n+1}, y_{n+1} such that $d(x_{n+1}, y_{n+1}) < \delta_{n+1}$ and $d(f(x_{n+1}), f(y_{n+1})) \geq 4\epsilon$. This completes the construction of the sequences $\{\delta_n\}, \{x_n\}, \{y_n\}$. We note that if $A = \{x_n : n \geq 1\}$ and $B = \{y_n : n \geq 1\}$ then $d(A, B) = 0$. We get the desired contradiction by showing that $d(f(A), f(B)) \geq \epsilon$. For this we have to show that $d(f(x_m), f(y_n)) \geq \epsilon$ for all m and n. For $m = n$ we already know that $d(f(x_n), f(y_m)) \geq 4\epsilon$. We first prove the inequality for $m < n$. Suppose, if possible, $d(f(x_m), f(y_n)) < \epsilon$ (*). Then $d(x_n, y_n) < \delta_n \leq \delta_{m+1} < d(N(x_m), F(x_m))$. Note that $y_n \in N(x_m)$ because $d(f(x_m), f(y_n)) < \epsilon$. If $x_n \in F(x_m)$ we would have $d(x_n, y_n) < d(N(x_m), F(x_m)) \leq d(y_n, x_n)$ a contradiction. Thus $x_n \notin F(x_m)$ which means $d(f(x_n), f(x_m)) < 2\epsilon$. But then $4\epsilon \leq d(f(x_n), f(y_n)) \leq d(f(x_n), f(x_m)) + d(f(x_m), f(y_n)) < 2\epsilon + \epsilon$ by (*). Now let $m > n$. Once again assume that $d(f(x_m), f(y_n)) < \epsilon$. Then $d(x_m, y_m) < \delta_m \leq \delta_{n+1} < d(N(y_n), F(y_n))$. Also the assumption that $d(f(x_m), f(y_n)) < \epsilon$ implies that $x_m \in N(y_n)$. Hence the previous inequality implies that $y_m \notin F(y_n)$. This means $d(f(y_m), f(y_n)) < 2\epsilon$. But then $4\epsilon \leq d(f(x_m), f(y_m)) \leq d(f(x_m), f(y_n)) + d(f(y_n), f(y_m)) < \epsilon + 2\epsilon$. This contradiction completes the proof.

[See also Problem 214 below]

Problem 87

An additive subgroup of $\mathbb{R}$ is either dense or discrete. There are additive subgroups which are dense and of first category and there are subgroups second category as well.

Let A be an additive subgroup of $\mathbb{R}$. If there is a sequence $\{t_n\} \subset A$ such that $t_n > 0 \ \forall n$ and $t_n \to 0$ as $n \to \infty$ then any interval $(a, b) \subset (0, \infty)$ has non-empty intersection with A. This is because the length of the interval $(\frac{a}{t_n}, \frac{b}{t_n})$ exceeds 1 if n is sufficiently large and hence it contains an integer m. It follows that $mt_n \in A \cap (a, b)$. This proves that A is dense in $(0, \infty)$. Since $-a \in A \ \forall a \in A$ it follows that A is dense in $\mathbb{R}$. In the contrary case let $\alpha = \inf(A \cap (0, \infty))$. If $\alpha \notin A$ then there is a sequence $\{a_n\}$ in A strictly decresing to α. In tis case $|a_n - a_m| \in (0, \alpha) \cap A$ for some n and m. Since this contradicts the definition of α it follows that $\alpha \in A$. If $x \in A \cap (\alpha, 2\alpha)$ then $x - a \in (0, \alpha) \cap A$ which is again a contradiction. Thus $A \cap (\alpha, 2\alpha) = \emptyset$. It follows by induction that $A \cap (n\alpha, (n+1)\alpha) = \emptyset$ for each positive integer n proving that $A \cap (0, \infty) = \{\alpha, 2\alpha, 3\alpha, \ldots\}$. Hence $A = \{n\alpha : n \in \mathbb{Z}\}$. Now let B be a basis for $\mathbb{R}$ over $\mathbb{Q}$ and let $\{b_n\}$ be a sequence of distinct points in B. Let A_n be the subgroup of $\mathbb{R}$ generated by $B \backslash \{b_{n+1}, b_{n+2}, ...\}$. Then $\mathbb{R} = \cup A_n$ so at least one A_n must be of second category. Since this group is not discrete it is dense.

Problem 88

Characterize metric spaces (X, d) such that pointwise convergence of a sequence real continuous functions on X implies uniform convergence.

Let $f_n(x) = \{\frac{1}{1+d(x_0,x)}\}^n$ where $x_0 \in X$ is fixed. Then $f_n(x) \to 0$ unless $d(x_0, x) = 0$ in which case $f_n(x) = 1 \ \forall n$. Thus, if (X, d) has the stated property then $I_{\{x_0\}}$ is continuous. In other words, $\{x_0\}$ is open and this is true for each x_0. If X is an infinite set with distinct elements $x_1, x_2, ...$ then $g_n(x) = \frac{k}{n}$ if $x = x_k, 0$ if $x \notin \{x_1, x_2, ...\}$ defines a sequence of continuous functions converging pointwise but not uniformly. Hence X is a finite set. The converse also holds.

Problem 89

Let $f : \mathbb{R} \to \mathbb{R}$ map intervals to intervals. Does it follow that f is continuous? What if f is also one-to-one?

$f(x) = \sin(\frac{1}{x})$ for $x \neq 0, 0$ for $x = 0$ maps intervals to intervals but it is not continuous. [Ramark: any derivative has intermediate value property(Problem 416 below and its solution?) but a derivative need not be continuous]. Now assume that f is one-to-one and has intermediate value property. We claim that f is montone. Once this is proved it follows easily that f is continuous. Suppose, if possible, $x < y < z$, $f(x) < f(y)$ and $f(z) < f(y)$. Let $I_1 = f([x, y])$ and $I_2 = f([y, z])$. Then I_1 and I_2 are intervals and $I_1 \cap I_2 = \{f(y)\}$. Let $[a_1, b_1]$ and $[a_2, b_2]$ be their closures. Either $b_2 = a_1$ or $b_1 = a_2$. Note that $f(z) < f(y) \leq b_1$ and $f(z) \geq a_2$ so $a_2 < b_1$. Thus we must have $b_2 = a_1$ and

$f(y) = b_2 = a_1$. But $f(x) \in I_1$ so $f(x) \geq a_1 = f(y)$ which is a contradiction. Thus we cannot have points x, y, z with $x < y < z$, $f(x) < f(y)$ and $f(z) < f(y)$. Similarly we cannot have $x < y < z$, $f(x) > f(y)$ and $f(z) > f(y)$. This proves that f is monotone.

Remark: the fact that f is one-to-one was used only to conclude that $I_1 \cap I_2 = \{f(y)\}$. This would be true if we only knew that $I_1 \cap I_2$ is a finite set. Thus if $f^{-1}(\{a\})$ is empty or a finite set for each a and if f has intermediate value property then it is continuous.

Problem 90

Let $f : (0, \infty) \to (0, \infty)$ be a convex function and $a, b \in \mathbb{R}$. Show that $xf(a + \frac{b}{x})$ is a convex function on $(0, \infty)$.

We can write $f(x) = \sup\{\alpha_i x + \beta_i : i \in I\}$. We have $xf(a + \frac{b}{x}) = \sup\{(a\alpha_i + \beta_i)x + b\alpha_i : i \in I\}$. Q.E.D.

Problem 91

Let A, B, C be subsets of a normed linear space X such that $A + C \subset B + C$ and C is bounded. Show that A is contained in the closed convex hull of B

If not, there exists $a_0 \in A$ and $x^* \in X^*$ such that $x^*(a_0) = 1$ and $x^*(x) < 0$ for all $x \in B$. Let $c \in C$ be such that $x^*(c) > \sup\{x^*(y) : y \in C\} - \epsilon$. There exists $u \in C$ and $b \in B$ such that $a_0 + c = b + u$. We have $1 + x^*(c) < x^*(u) \leq \sup\{x^*(y) : y \in C\} < x^*(c) + \epsilon$ which is a contradiction.

Problem 92

Let A, B, C, D be $n \times n$ matricies such that $AD^* - BC^* = I, AB^* = BA^*$ and $CD^* = DC^*$. Prove that $A^*D - C^*B = I$.

We have $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix} = I$ and hence $\begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = I$. This implies that $A^*D - C^*B = I$.

Problem 93

Let (X, d) be a metric space such that for any $x_1, x_2 \in X$ there exists $u \in X$ with $d^2(x_1, x_2) + 4d^2(x, u) \leq 2d^2(x_1, x) + 2d^2(x_2, x)$ for all $x \in X$. Show that u is uniquely determined by x_1 and x_2 and that $d(u, x_1) = d(u, x_2) = \frac{1}{2}d(x_1, x_2)$. Prove or disprove that $d^2(x_1, x_2) + 4d^2(x, \frac{x_1+x_2}{2}) \leq 2d^2(x_1, x) + 2d^2(x_2, x)$ for all $x \in X$ when X is a normed linear space.

We have $d^2(x_1,x_2)+4d^2(x_1,u)\le 2d^2(x_1,x_2)$ and $d^2(x_1,x_2)+4d^2(x_2,u)\le 2d^2(x_1,x_2)$ so $d^2(x_1,u)\le \frac{1}{4}d^2(x_1,x_2)$ or $d(x_1,u)\le \frac{1}{2}d(x_1,x_2)$ and $d(x_2,u)\le \frac{1}{2}d(x_1,x_2)$. If strict inequality holds in one of these we get $d(x_1,x_2)<\frac{1}{2}d(x_1,x_2)+\frac{1}{2}d(x_1,x_2)=d(x_1,x_2)$, a contradiction. Hence $d(u,x_1)=d(u,x_2)=\frac{1}{2}d(x_1,x_2)$. Now $4d^2(x,u)\le 2d^2(x_1,x)+2d^2(x_2,x)-d^2(x_1,x_2)\ \forall x$. If x is a point of x with $d(x,x_1)=d(x,x_2)=\frac{1}{2}d(x_1,x_2)$ we get $4d^2(x,u)\le 0$ so $x=u$. If X is a normed linear space then the inequality $d^2(x_1,x_2)+4d^2(x,\frac{x_1+x_2}{2})\le 2d^2(x_1,x)+2d^2(x_2,x)$ need not hold for all $x\in X$! Let $X=C[0,1], x_1(t)=0, x_2(t)=2-2t$ $(0\le t\le 1)$. If $x(t)=1(0\le t\le 1)$ then $d(x_1,x_2)=2, d(x,\frac{x_1+x_2}{2})=1, d(x_1,x)=1$ and $d(x_2,x)=1$. Hence $d^2(x_1,x_2)+4d^2(x,\frac{x_1+x_2}{2})=8$ and $2d^2(x_1,x)+2d^2(x_2,x)=4$.

Problem 94

True or false: if X is a normed linear space then $\|x-y\|^2+\|x+y\|^2\le 2\|x\|^2+2\|y\|^2\ \forall x,y\in X$.
True or false: if X is a normed linear space then $\|x-y\|^2+\|x+y\|^2\ge 2\|x\|^2+2\|y\|^2\ \forall x,y\in X$.

Both are false. In fact the transformation $(x,y)\to(\frac{x+y}{2},\frac{x-y}{2})$ shows that the two properties are equivalent and hence they are both equivalent to the fact that X is an inner product space.

Problem 95

Let X be a normed linear space and $f: X\to\mathbb{R}$ is *locally convex* in the sense for each $x\in X$ there exists $\delta>0$ such that f is convex on $B(x,\delta)$. Does it follow that f is convex on X?

Yes. Let $x,y\in X$ and consider the function $g(t)=f(tx+(1-t)y), 0\le t\le 1$. Then g is locally convex on $[0,1]$. Hence its right hand derivative is *locally* increasing which implies it is increasing. Hence g is a convex function and so is f. [$g(t)\le(1-t)g(0)+tg(1)$].

Problem 96

Let $\{\phi_n\}$ be a sequence of continuous functions : $(0,\infty)\to(0,\infty)$. Show that there is a continuous function $f:(0,\infty)\to(0,\infty)$ which $\to\infty$ faster then each of the $\phi_n' s$ [i.e. $\lim\limits_{x\to\infty}\frac{f(x)}{\phi_n(x)}=\infty$ for each n]

Let $a_n=\sup\{f_n(x): 0<x\le n\}$ where $f_n=\max\{\phi_1(x),\phi_2(x),...,\phi_n(x)$.
Let $f(x)=\begin{cases} (n+1)a_{n+1}+1 \text{ if } n\le x\le n+1/2 \\ 2[(n+2)a_{n+2}-(n+1)a_{n+1}]x+(n+1)a_{n+1}+1-(2n+1)[(n+2)a_{n+2}-(n+1)a_{n+1}] \text{ if } n+1 \end{cases}$

Then f is continuous and $\inf_{n\le x\le n+1} f(x) = (n+1)a_{n+1}+1$. It follows that $\frac{\inf_{n\le x\le n+1} f(x)}{\sup_{n\le x\le n+1} f_n(x)} \ge \frac{\inf_{n\le x\le n+1} f(x)}{\sup_{0<x\le n+1} f_n(x)} = \frac{\inf_{n\le x\le n+1} f(x)}{a_{n+1}} = \frac{(n+1)a_{n+1}+1}{a_{n+1}} > (n+1)$. Thus $\frac{f(x)}{f_n(x)} > n+1$ if $n \le x \le n+1$. If $n+k \le x \le n+k+1$ then $\frac{f(x)}{f_n(x)} \ge \frac{f(x)}{f_{n+k}(x)} > n+k$. This implies that $\frac{f(x)}{f_n(x)} \to \infty$ as $x \to \infty$ for each n. Of course, $\frac{f(x)}{\phi_n(x)} \ge \frac{f(x)}{f_n(x)}$ so $\frac{f(x)}{\phi_n(x)} \to \infty$ as $x \to \infty$ for each n.

Remark: $\sup\{f(y) : 0 \le y \le x\}$ is easily seen to be a continuous increasing function exceeding f at every point. By problem 197 of CASolutions.tex it follows that there is an entire function g such that $g(x) \ge \phi_n(x)\ \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$. In particular g is "smooth".

Problem 97

Prove that the Hausdorff dimension of the Cantor's ternary set C is $\frac{\log 2}{\log 3}$

[For $A \subset \mathbb{R}$ and $p > 0$ let $\mu_p(A) = \lim_{\epsilon\to 0} \inf\{\sum_{n=1}^{\infty}(diam(U_n))^p : U_n's$ are open with $diam(U_n) \le \epsilon\ \forall n$ and $A \subset \bigcup_{n=1}^{\infty} A_n\}$. There is a unique $d > 0$ such that $\mu_p(A) = \infty$ if $p < d$ and $\mu_p(A) = 0$ if $p > d$. d Is called the Hausdorff dimension of A].

Let $\mu_{p,\epsilon}(A) = \inf\{\sum_{n=1}^{\infty}(diam(U_n))^p : U_n's$ are open with $diam(U_n) \le \epsilon\ \forall n$ and $A \subset \bigcup_{n=1}^{\infty} U_n\}$. Let the 2^n closed intervals that remain at the $n-th$ stage in the construction of C be $I_{n,1}, I_{n,2}, ..., I_{n,2^n}$. Then $C \subset \bigcup_{j=1}^{2^n} I_{n,j}$ and $\mu_{p,3^{-n}}(C) \le \sum_{j=1}^{2^n}(diam(I_{n,j}))^p = 2^n\frac{1}{3^{np}} = 1$ if $p = \frac{\log 2}{\log 3}$. Hence the Huasdorff dimension of C does not exceed $\frac{\log 2}{\log 3}$.

Now let $U_n's$ be bunded open sets with and $C \subset \bigcup_{n=1}^{\infty} U_n$. Let $J_n = [\inf U_n, \sup U_n]$. Let $V_n = (\inf U_n - \frac{\epsilon}{2^n}, \sup U_n + \frac{\epsilon}{2^n})$. Then $C \subset \bigcup_{n=1}^{\infty} V_n$ and hence there is a positive integer N such that $C \subset \bigcup_{n=1}^{N} V_n$. If $diam(V_n) < 1/3$ then we can find an integer k_n such that $\frac{1}{3^{k_n+1}} \le diam(V_n) < \frac{1}{3^{k_n}}$. Note that $\bar{V_n}$ can intersect at most one of the intervals $I_{k_n,1}, I_{k_n,2}, ..., I_{k_n,2^{k_n}}$. This is because these intervals are separated by a distance of $\frac{1}{3^{k_n}}$ and $diam(V_n) < \frac{1}{3^{k_n}}$. Choose j so large that $\frac{1}{3^{j+1}} \leq diam(\bar{V_n})$ and $j > k_n$ for $1 \leq n \leq N$. If $\bar{V_n}$ intersects one of the intervals $I_{k_n,1}, I_{k_n,2}, ..., I_{k_n,2^{k_n}}$, say $I_{k_n,l}$ then $I_{k_n,l}$ contains 2^{j-k_n} intervals at the $j-th$ step in the construction of C. Hence $\bar{V_n}$ intersects at most 2^{j-k_n} intervals at the $j-th$ step in the construction of C. Hence, the sets $\bar{V_n}, 1 \leq n \leq N$ can intersect atmost $\sum\limits_{n=1}^{N} 2^{j-k_n}$ of those intervals. But $C \subset \bigcup\limits_{n=1}^{N} \bar{V_n}$ and hence all the intervals $I_{j,1}, I_{j,2}, ..., I_{j,2^j}$ intersect $\bigcup\limits_{n=1}^{N} \bar{V_n}$. It follows that if $t = \frac{\log 2}{\log 3}$ then $2^j \leq \sum\limits_{n=1}^{N} 2^{j-k_n} \leq \sum\limits_{n=1}^{N} 2^{j-k_n} 3^{t+k_n t}(diam V_n))^t$ [because $\frac{1}{3^{k_n+1}} \leq diam(V_n)] = \sum\limits_{n=1}^{N} 2^{j+1}(diam V_n))^t$ and $\sum\limits_{n=1}^{N}(diam V_n))^t \geq \frac{1}{2} = 3^{-t}$. This gives $\sum\limits_{n=1}^{N}(diam U_n) + \epsilon/2^{n-1})^t \geq 3^{-t}$ and $\sum\limits_{n=1}^{\infty}(diam U_n) + \epsilon/2^{n-1})^t \geq 3^{-t}$. Letting $\epsilon \to 0$ we get $\sum\limits_{n=1}^{\infty}(diam U_n))^t \geq 3^{-t}$. This holds for any cover of C by bounded open sets and hence $\mu_t(C) \geq 3^{-t}$. In particular $\mu_t(C) > 0$ and hence the Hausdorff dimension of C is at least $t = \frac{\log 2}{\log 3}$. This completes the proof.

Problem 98

Show that there is no sequence $\{a_n\}$ converging to 0 such that $\hat{f}(n) \to 0$ faster then $\{a_n\}$ for every continuous function f on $\mathbb{R}$ with period 2π.["$\hat{f}(n) \to 0$ faster then $\{a_n\}$" means $\frac{\hat{f}(n)}{a_n} \to 0$].

For each k let n_k be so large that $|a_{n_k}| < \frac{1}{k^2}$. We may suppose $n_k < n_{k+1}$ $\forall k$. Let $f(x) = \sum\limits_{k=1}^{\infty} \frac{e^{in_k x}}{k^2}$. Then $\left|\frac{\hat{f}(n_k)}{a_{n_k}}\right| = \left|\frac{1}{k^2 a_{n_k}}\right| > 1$

Problem 99

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim\limits_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = 0$ $\forall x$. Prove that $\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = 0$ for all x and all $h \in \mathbb{R}$. Find all functions f with this property.

If $f(x) \equiv \alpha x + \beta$ then $\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \frac{\alpha(x+h)+\alpha(x-h)-2\alpha x}{h^2} = 0$. We prove that the only functions satisfying the given property are these. Let $g(x) =$

$f(x)-f(a)-\frac{x-a}{b-a}\{f(b)-f(a)\}$ for $a<x<b$ (where a and b with $a<b$ are arbitrary).

Then $g(a)=0=g(b)$. If we prove that $g(x)=0\ \forall x\in(a,b)$ then, since a and b are arbitrary we get the desired conclusion. Suppose, if possible, $g(x_0)>0$ for some $x_0\in(a,b)$. Let $\phi(x)=g(x)-\frac{1}{2}\delta(x-a)(b-x)$ where $\delta>0$ is small that $g(x_0)-\frac{1}{2}\delta(x_0-a)(b-x_0)>0$. Then ϕ has a positive maximum on $[a,b]$ attained at some point u of (a,b). We have $\phi(u-h)+\phi(u+h)-2\phi(u)\leq 0$ for $|h|$ sufficiently small. We now compute $\frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{h^2}$ in terms of g. We get $\frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{h^2}=\frac{g(x+h)+g(x-h)-2g(x)}{h^2}-\frac{\delta}{2}\frac{(x+h-a)(b-x-h)+(x-h-a)(b-x+h)-2(x-a)(b-x)}{h^2}=\frac{g(x+h)+g(x-h)-2g(x)}{h^2}+\delta=\frac{f(x+h)+f(x-h)-2f(x)}{h^2}+\delta\to\delta$ as $h\to 0$. This contradiction shows that $f(x)\equiv\alpha x+\beta$ for some α and β.

Problem 100

Let $\{a_n\}$ be a sequence of real numbers such that $\sum\limits_{n=0}^{\infty}a_nx^n$ converges for all $x>0$. Show that the equation $\int\limits_0^{\infty}e^{-x}\sum\limits_{n=0}^{\infty}a_nx^n dx=\sum\limits_{n=0}^{\infty}a_n\int\limits_0^{\infty}e^{-x}x^n dx$ holds if the series on the right is convergent.

Remark: this is a result on interchange of limit and integral where the basic theorems of measure theory don't seem to be of much use!

First note that $\int\limits_0^{\infty}e^{-x}x^n dx=n!$. Let $b_n=(n!)a_n$. It is given that $\sum\limits_{n=0}^{\infty}b_n$ is convergent. Let $\epsilon>0$ and choose $T\in(0,\infty)$ such that $e^{-T}\sum\limits_{n=0}^{N}\frac{T^n}{n!}<\epsilon$ where N is so large that $\left|\sum\limits_{n=k}^{\infty}b_n\right|<\epsilon$ for $k>N$. We write c_k for $\sum\limits_{n=k}^{\infty}b_n$ so $|c_k|<\epsilon$ for $k>N$.

Now $\left|e^{-T}\sum\limits_{n=0}^{\infty}c_n\frac{T^n}{n!}\right|\leq(\sup|c_n|)\ e^{-T}\sum\limits_{n=0}^{N}\frac{T^n}{n!}+\epsilon e^{-T}\sum\limits_{n=N+1}^{\infty}\frac{T^n}{n!}<(\sup|c_n|)$ $e^{-T}\sum\limits_{n=0}^{N}\frac{T^n}{n!}+\epsilon<\epsilon(1+\sup|c_n|)$. We have to show that $\sum\limits_{n=0}^{\infty}\frac{b_n}{n!}\int\limits_T^{\infty}e^{-x}x^n dx\to 0$ as $T\to\infty$. [Indeed

$\int\limits_0^{T}\sum\limits_{n=0}^{\infty}a_ne^{-x}x^n dx=\sum\limits_{n=0}^{\infty}a_n\int\limits_0^{T}e^{-x}x^n dx$ (by uniform convergence of the power series)

$$= \sum_{n=0}^{\infty} a_n(\int_0^{\infty} e^{-x}x^n dx - \int_T^{\infty} e^{-x}x^n dx) = \sum_{n=0}^{\infty} a_n(\int_0^{\infty} e^{-x}x^n dx) - \sum_{n=0}^{\infty} a_n(\int_T^{\infty} e^{-x}x^n dx)].$$

Note that $\int_T^{\infty} e^{-x}x^n dx = e^{-T}\{T^n + nT^{n-1} + ... + (n!)T^0\}$. To show $\sum_{n=0}^{\infty} \frac{b_n}{n!} e^{-T}\{T^n + nT^{n-1} + ... + (n!)T^0\} \to 0$. This means $\sum_{n=0}^{\infty} b_n e^{-T}\{\frac{T^n}{n!} + \frac{T^{n-1}}{(n-1)!} + ... + \frac{T}{1!} + T^0\} \to 0$. Writing b_n as $c_n - c_{n+1}$ this becomes $e^{-T}\sum_{n=0}^{\infty} c_n \frac{T^n}{n!}$. We have already seen that $\left|e^{-T}\sum_{n=0}^{\infty} c_n \frac{T^n}{n!}\right| < \epsilon(1 + \sup|c_n|)$ for T such that $e^{-T}\sum_{n=0}^{N} \frac{T^n}{n!} < \epsilon$ where N is so large that $\left|\sum_{n=k}^{\infty} b_n\right| < \epsilon$ for $k > N$ and the proof is complete.

Problem 101

If the graph of $f : \mathbb{R} \to \mathbb{R}$ is closed and connected then f is continuous. This does not extend to maps between general connected metric spaces.

First, the counter-example: let X be $C[0,1]$ with the L^1 metric and Y be $C[0,1]$ with the sup metric. Let f be the identity map from X to Y. The graph of this map is a subspace, hence convex, hence path connected. The graph is clearly closed but f is not continuous. Now the proof of the first part: Let $x_n \to x$ and $|f(x_n)| \to \infty$. We claim that there is a $\delta > 0$ such that $|y - x| \leq \delta \Rightarrow |f(x) - f(y)| < 1$ or $|f(x) - f(y)| > 2$. If the claim is false then we can find a sequence $\{u_n\}$ converging to x such that $1 \leq |f(x) - f(u_n)| \leq 2$ $\forall n$. There is a subsequence $\{f(u_{n_k})\}$ of $\{f(u_n)\}$ converging to some point w. Since the graph is closed we get $w = f(x)$. But $1 \leq |f(x) - w|$. This proves the claim. Let G be the graph of f. Then $G \cap \{[a,b] \times \mathbb{R}\} = (G \cap \{[a,b] \times \mathbb{R}\} \cap \{(t,s) : |f(x) - s| < 1\}) \cup (G \cap \{[a,b] \times \mathbb{R}\} \cap \{(t,s) : |f(x) - s| > 1\})$ where $[a,b]$ is the interval $[x - \delta, x + \delta]$. If we prove that $G \cap \{[a,b] \times \mathbb{R}\}$ is connected we get a contradiction because the two sets on the right contain $(x, f(x))$ and $(x_n, f(x_n))$ for n sufficiently large. This would prove that $x_n \to x$ implies that $\{f(x_n)\}$ is bounded and the fact that g is closed shows the only limit point of this bounded sequence is $f(x)$. It follows that f is continuous. To complete the proof we prove that $G \cap \{[a,b] \times \mathbb{R}\}$ is connected. If $g : G \cap \{[a,b] \times \mathbb{R}\} \to \{0,1\}$ is continuous we can extend it to a continuous function $g : G \to \{0,1\}$ by making it constant on $g : G \cap \{(-\infty, a] \times \mathbb{R}\}$ and on $g : G \cap \{[b, \infty) \times \mathbb{R}\}$. The extended function must be a constant and so must be the original function.

Problem 102.

Let $I = (a, b)$ be a finite or infinite open interval in $\mathbb{R}$ and d be a metric on it which is equivalent to the usual metric. Prove that there exist disjoint closed sets A and B in I such that $d(A, B) = 0$.

If $a = -\infty$ and $b = \infty$ let $A = \mathbb{N}$. For each $n \in \mathbb{N}$ choose $\delta_n > 0$ such that $d(n, n + \delta_n) < \frac{1}{n}$. This is possible because $n + \frac{1}{k} \to n$ as $k \to \infty$. Let $B = \{n + \delta_n : n \in \mathbb{N}\}$. Then A and B are disjoint closed sets in (I, d) and $d(A, B) = 0$.

If $a > -\infty$ take $A = \{a + \frac{1}{n} : n \in \mathbb{N}\}$ and $B = \{a + \frac{1}{n} + \delta_n : n \in \mathbb{N}\}$ where $d(a + \frac{1}{n}, a + \frac{1}{n} + \delta_n) < \frac{1}{n}\ \forall n$. If $b < \infty$ take $A = \{b - \frac{1}{n} : n \in \mathbb{N}\}$ and $B = \{b - \frac{1}{n} - \delta_n : n \in \mathbb{N}\}$ where $d(b - \frac{1}{n}, b - \frac{1}{n} - \delta_n) < \frac{1}{n}\ \forall n$.

Problem 103

Suppose $A \subset \mathbb{R}^n$ is such that the distance between any two points is rational. Prove that A is atmost countable.

By translation we may suppose $0 \in A$. The result is obvious for $n = 1$ since $A \subset Q$ in that case. Assume that the result holds in $\mathbb{R}^k$ if $k < n$. We may assume that A spans $\mathbb{R}^n$. Let $\{x_1, x_2, ..., x_n\}$ be a basis for $\mathbb{R}^n$ contained in A. For any rationals $r, r_1, ..., r_n$ we claim that there is at most one point x such that $\|x - x_i\| = r_i\ \forall i$ and $\|x\| = r$. Indeed if x and y both have norm r and distance r_i from $x_i = x_i\ \forall i$ then $< x, x_i >= \frac{1}{2}[r_i^2 - r^2 - \|x_i\|^2] =< y, x_i >$ $\forall i$ which means $x - y$ is orthogonal to each x_i. Thus, with each $a \in A$ we can associate $(n+1)$ rational numbers $r, r_1, ..., r_n$ and this association is one-to-one. Note that $\mathbb{Q}$ can be replaced by any countable set.

Problem 104

Let $A \subset \mathbb{R}^n$ be countable. Show that $\mathbb{R}^n \backslash A$ is connected.

Consider the sets $\{tx : t > 0\}$ where $\|x\| = 1$. These sets are disjoint and hence only countable many of them can intersect A. Similarly $\{y : \|x\| = r\}$ can intersect A for at most countably many positive numbers r. Removing these we get rays and circles disjoint from A and the union of these rays and circles is a connected dense subset of $\mathbb{R}^n$ containes in $\mathbb{R}^n \backslash A$. Since any set that lies between a connected set and its closure is connected the result follows.

Problem 105

Let X be a separable normed linear space and f be a continuous linear functional on a subspace M of X. Show without using Zorn's Lemma (or any of its equivalents) that f can be extended to a continuous linear functional on X with the same norm.

Let $\{x_n\}$ be the intersection with M^c of a countable dense set in X. Let $M_n = span(M \cup \{x_1, x_2, ..., x_n\}), n = 1, 2, ...$ As in the usual proof of Hahn-Banach

Theorem we get extensions $f_1, f_2, ...$ of f to $M_1, M_2, ...$ such that $\|f_n\| = \|f\| \; \forall n$ and $f_{n+1} = f_n$ on $M_n \; \forall n$. Let $N = \bigcup\limits_{n=1}^{\infty} M_n$ and define $g(x) = f_n(x)$ if $x \in M_n$. Then g is a continuous linear map on the subspace N and $\|g(x)\| \leq \|f\| \, \|x\|$ $\forall x$. Since N is dense in X it is obvious that g extends to a continuous linear functional on X with the same norm as f.

Problem 106

Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(x) > \int\limits_0^x f(t)dt$ $\forall t \in [0,1]$. Prove that $f(x) > 0 \; \forall x \in [0,1]$.

Is the following discrete analog true?

If $a_1, a_2, ..., a_N$ are real numbers such that $a_{k+1} > a_1 + a_2 + ... + a_k$ for $1 \leq k < n$ then $a_k > 0$ for all k.

Let $g(x) = e^{-x} \int\limits_0^x f(t)dt$. Then $g'(x) = e^{-x}[f(x) - \int\limits_0^x f(t)dt] > 0$ for all x. Hence g is strictly increasing. Also $g(0) = 0$ so $g(x)$ is positive. It follows that g^2 is also strictly increasing. Now $\frac{d}{dx} g^2(x) = 2g(x)g'(x) = \{2e^{-x} \int\limits_0^x f(t)dt\}\{e^{-x}[f(x) - \int\limits_0^x f(t)dt]\}$. This proves that $\int\limits_0^x f(t)dt \geq 0$ and the hypothesis shows $f(x) > 0$. The discrete version is obviously false. [And obviously $a_k > 0$ for $k > 1$ if $a_1 = 0$].

Problem 107

Let $p(x) = x^2 + ax + b$ and A be the 3×3 matrix with entries $p(i-j), 0 \leq i, j \leq 2$. Show that the determinant of A does not depend on the coefficients of p.

We have $A = \begin{pmatrix} p(0) & p(-1) & p(-2) \\ p(1) & p(0) & p(-1) \\ p(2) & p(1) & p(0) \end{pmatrix}$. Add the first column and (-2) times the second column to the third column. One sees easily that the third column becomes $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$. For example, $p(0) - 2p(-1) + p(-2) = b - 2(1 - a + b) + (4 - 2a + b) = 2$. Now subtract the first row from the second and third rows to get a matrix whose last column is $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$. Expanding the determinant from the last column we see that the value of the determinant is $2[p(1)-p(0)][p(1)-p(-1)]-2[p(2)-p(0)][[p(0)-p(-1)]$ which is 8.

Remarks: the argument actually works for monic polynomials of any degree and the value of the determinant is $(n!)^{\ n+1}$ when $p(x)$ is of the type $x^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_1x+a_0$. [AMM, 2010]

Problem 108

Let A be a bounded set in a Hilbert space. Show that there is a unique closed ball of minimal radius containing A.

Let $\alpha=\inf\{r>0 : A\subset \bar{B}(x,r)$ for some x. Let $\epsilon>0$ and $A\subset \bar{B}(x_n,r_n)$ with $r_n\downarrow\alpha$. Let n_0 be such that $r_n<\sqrt{\alpha^2+\epsilon}$ for $n\geq n_0$. Claim: $A\subset \bar{B}(\frac{x_n+x_m}{2},\sqrt{\alpha^2-\epsilon})$ if n and m are $\geq n_0$ and $\|x_n-x_m\|\geq 8\epsilon$. Once this claim is established we get a contradiction to the definition of α and we can conclude that $\|x_n-x_m\|<8\epsilon$ whenever n and m are $\geq n_0$. This would prove that $\{x_n\}$ is Cauchy; if $x_n\to x$ it is clear that $A\subset \bar{B}(x,\alpha)$ proving the existence part. Let $a\in A$. Then $\|a-x_n\|\leq r_n$ and $\|a-x_m\|\leq r_m$. This gives $\|x_n-x_m\|^2+4\left\|a-\frac{x_n+x_m}{2}\right\|^2=2\|a-x_n\|^2+2\|a-x_m\|^2\leq 2r_n^2+2r_m^2<4(\alpha^2+\epsilon)$. Hence $\|x_n-x_m\|\geq 8\epsilon$ implies $4\left\|a-\frac{x_n+x_m}{2}\right\|^2\leq 4(\alpha^2+\epsilon)-8\epsilon$. This holds for all $a\in A$ so $A\subset \bar{B}(\frac{x_n+x_m}{2},\sqrt{\alpha^2-\epsilon})$. This completes the proof of existence. Uniqueness: suppose $A\subset \bar{B}(x,\alpha)$ and $A\subset \bar{B}(y,\alpha)$. If $a\in A$ then $\|a-x\|\leq\alpha$ and $\|a-y\|\leq\alpha$. We have $\|x-y\|^2+4\left\|a-\frac{x+y}{2}\right\|^2=2\|a-x\|^2+2\|a-y\|^2\leq 4\alpha^2$. If $\|x-y\|=\delta>0$ then $\left\|a-\frac{x+y}{2}\right\|^2\leq\alpha^2-\frac{1}{4}\delta^2$ $\forall a\in A$ contradicting the definition of α. Hence $x=y$.

Problem 109 [See also Problem 1]

Let μ be a finite positive measure on the Borel subsets of $(0,\infty)$. If $g\in L^\infty(\mu)$ and $\int_0^\infty e^{-x}p(x)g(x)d\mu(x)=0$ for every polynomial p show that $g=0$ a.e. $[\mu]$. Conclude that $\{e^{-x}p(x) : p$ is a polynomial$\}$ is dense in $L^1(\mu)$.

The second part follows immediately from the first. For the first part let $\phi(z)=\int_0^\infty e^{-zx}g(x)d\mu(x)$ for $z\in\mathbb{C}$ with $\mathrm{Re}(z)>0$. A straightforward argument shows that ϕ is analytic in $\{z\in\mathbb{C}\text{: } \mathrm{Re}(z)>0\}$. Further, $\phi^{(n)}(z)=\int_0^\infty(-x)^ne^{-zx}g(x)d\mu(x)$ for $z\in\mathbb{C}$ and $n\geq 0$. By hypothesis this gives $\phi^{(n)}(1)=0$

$\forall n \geq 0$. It follows that $\phi(z) = 0$ whenever $\mathrm{Re}(z) > 0$. In particular $\int\limits_0^\infty e^{-tx} g(x) d\mu(x) = 0$ if $t > 0$. The finite positive measures ν_1 and ν_2 defined by $d\nu_1 = g^+ d\mu$ and $d\nu_2 = g^- d\mu$ have the same Laplace transform and hence they are equal. This means $g(x)d\mu(x) = 0$ which is what we wanted to prove.

Problem 110

Find all continuous functions $f : (0, \infty) \to (0, \infty)$ such that $x \to \int\limits_x^{kx} f(t)dt$ is constant on $(0, \infty)$.

If $k = 1$ then f is arbitrary. Assume that $k \neq 1$. We have $0 = \frac{d}{dx} \int\limits_x^{kx} f(t)dt = kf(kx) - f(x)$. Let $g(x) = xf(x)$. Then g is continuous on $(0, \infty)$ and $g(kx) = kxf(kx) = xf(x) = g(x)$ $\forall x$. There is a continuos function h on $(0, \infty)$ such that $h(x + \alpha) = h(x)$ $\forall x$ and $f(x) = \frac{g(x)}{x} = \frac{h(\ln x)}{x}$ where $\alpha = \ln k$. This is the desired characterization.

Problem 111

Let $A \subset \mathbb{C}$ be a convex set such that $x \in A \Rightarrow -x \in A$. If $a_1, a_2, a_3 \in A$ show that at least one of the 6 numbers $a_1 + a_2, a_1 - a_2, a_2 + a_3, a_2 - a_3, a_3 + a_1, a_3 - a_1$ must be in A.

Since the three given points are necessarily lineraly dependent over $\mathbb{R}$ we can find $r, s, t \in \mathbb{R}$ not all 0 such that $ra_1 + sa_2 + ta_3 = 0$. Suppose $t = 0$. Then we can either write $a_1 = \lambda a_2$ or $a_2 = \lambda a_1$. If $a_1 = \lambda a_2$ it is easy to see that $a_2 - a_1 \in A$. By symmetry the same thing holds if $a_2 = \lambda a_1$. Similar argument can be given in the case $s = 0$ and the case $r = 0$. Assume now that all the coefficients r, s, t are non-zero. By a suitable change of notations we may suppose $|r| \leq |s|$ and $|r| \leq |t|$. We have $a_1 = -\frac{s}{r} a_2 - \frac{t}{r} a_3$. Since

$x \in A \Rightarrow -x \in A$ we may suppose $a_1 = \alpha a_2 + \beta a_3$ with α and $\beta \geq 1$. Now $a_2 + a_3 = \rho_1 a_1 + \rho_2 a_2 + \rho_3 a_3$ where $\rho_1 = \frac{1}{\alpha+\beta-1}, \rho_2 = \frac{\beta-1}{\alpha+\beta-1}, \rho_3 = \frac{\alpha-1}{\alpha+\beta-1}$. This proves that $a_2 + a_3 \in A$.

Problem 112

Show that every polynomial p with real coefficients and real roots satisfies the inequality $(n-1)[p'(x)]^2 \geq np(x)[p''(x)]$ where n is the degree of p.

We use induction on n. The result is obvious if $n = 1$. Assume that it holds for $n = k$ and consider a polynomial p of degree $k + 1$ with real coefficients and real roots. We can write $p(x) = (x-a)q(x)$ where q is a polynomial of degree k and $a \in \mathbb{R}$. We have to show that $k[(x-a)q'(x)+q(x)]^2 \geq (k-1)(x-a)q(x)[(x-a)q''(x)+2q'(x)]$. By induction hypothesis the right side is $\leq (k-1)(x-a)^2\frac{k-1}{k}[q'(x)]^2 + 2(k-1)(x-a)q(x)q'(x)$. We have to show that $\frac{1}{k}(x-a)^2[q'(x)]^2 + k[q(x)]^2 \geq 2(x-a)q(x)q'(x)$. This inequality says that $[\frac{1}{\sqrt{k}}(x-a)q'(x) - \sqrt{k}q(x)]^2 \geq 0$ which is true.

Problem 113

Find $\sup\{\frac{(\int_0^1 f(x)dx)^2(\int_0^1 g(x)dx)^2}{\int_0^1 [f(x)]^2dx\int_0^1 [g(x)]^2dx} : f, g : [01] \to \mathbb{R}$ are continuous, $\int_0^1 f(x)g(x)dx = 0\}$.

The answer is $\frac{1}{4}$. To show that the supremum is $\leq \frac{1}{4}$ we may suppose that $\int_0^1 [f(x)]^2dx = 1 = \int_0^1 [g(x)]^2dx$. We can extend $\{f, g\}$ to an orthonormal basis for $L^2([0,1])$ and Bessel's inequality gives $\int_0^1 1^2dx \geq (\int_0^1 f(x)dx)^2 + (\int_0^1 g(x)dx)^2$. Hence $1 \geq 2(\int_0^1 f(x)dx)(\int_0^1 g(x)dx)$ showing that the given spremum is $\leq \frac{1}{4}$. Let $f \in C([0,1])$ satisfy the conditions $\int_0^1 f(x)dx = 1$ and $\int_0^1 [f(x)]^2dx = 2$. $[1+\sqrt{3}(2x-1)$ satisfies these conditions]. Let $g = 2-f$. Then $\int_0^1 f(x)g(x)dx = 0$, $\int_0^1 g(x)dx = 1$ and $\int_0^1 [g(x)]^2dx = 2$. Hence $\frac{(\int_0^1 f(x)dx)^2(\int_0^1 g(x)dx)^2}{\int_0^1 [f(x)]^2dx\int_0^1 [g(x)]^2dx} = \frac{1}{4}$.

Problem 114

a) Let U be an open set in $\mathbb{R}$, F a closed set and $U \subset F$. Show that there is a set A whose interior is U and closure is F.

b) Find all sets $A \subset \mathbb{R}$ such that $A = \partial B$ for some $B \subset \mathbb{R}$.

Let $A = U \cup (\partial F) \cup ((Q \cap F) \backslash \bar{U})$. [$\partial F$ is the boundary of F]. $U \subset A$ so $U \subset A^0$. Let $x \in A^0$. Suppose, if possible, $x \notin U$. Note that $A \subset F$ so $A^0 \subset F^0$. Thus $x \in F^0$. Since $x \in A$ and $x \notin U \cup \partial F$ it follows that $x \in (Q \cap F) \backslash \bar{U}$. Thus, $x \notin \bar{U}$. We can find $r > 0$ such that $B(x, r) \subset A \cap F^0 \cap (U)^c \subset Q \cap F$ (by definition of A). But Q has no interior and this contradiction proves that $A^0 = U$. Now, $\bar{A} \subset F$. Let $x \in F$. If $x \in \partial F$ then $x \in A \subset \bar{A}$. Otherwise, $x \in F^0$. If $x \in \bar{U}$ then $x \in \bar{A}$. Otherwise $x \in F^0 \backslash \bar{U}$ and there exists $\delta > 0$ such that $B(x, \delta) \subset F^0 \backslash \bar{U}$. If $\{x_n\}$ is a sequence of rationals in $B(x, \delta)$ converging to x then $x_n \in (Q \cap F) \backslash \bar{U}$ for each n which implies $x_n \in A$ for each n. Thus $x = \lim x_n \in \bar{A}$. This completes the proof.

b) If A is any closed set in $\mathbb{R}$ let $D = \partial A \cup (Q \cap A^0)$. Clearly $A \subset \partial A \cup (Q \cap A^0)^- \subset \bar{D}$. Hence $\bar{D} = A$. We claim that $D^0 = \emptyset$. Suppose $x \in D^0$. If $x \in Q \cap A^0$ then there is a ball $B(x, r) \subset D \cap A^0$ and hence this ball cannot intersect ∂A. However $B(x, r) \subset D = \partial A \cup (Q \cap A^0)$ so $B(x, r) \subset D = (Q \cap A^0) \subset Q$, a contradiction. Thus, $x \notin Q \cap A^0$ which shows $x \in \partial A$. We have proved that $D^0 \subset \partial A$. Clearly ∂A has no interior and we have proved that $D^0 = \emptyset$. Now note that $\partial D = \bar{D} \backslash D^0 = A \backslash \emptyset = A$. Thus a set is the boundary od another set if and only if it is closed.

Remark: only two properties of Q are used in the proofs above: it has no interior and it is dense. The results therefore extend to any topological space in which such a set exists. [Countability of Q is not required]

Problem 115

Let H be a Hilbert space and C be a closed convex subset. For any $x \in H$ let Px be the unique point of C that is closest to C. Show that $\|x - y\|^2 \geq \|x - Px\|^2 + \|y - Px\|^2 \ \forall y \in C$.

We have $\|x - y\|^2 = \|x - Px\|^2 + \|y - Px\|^2 + 2 \operatorname{Re} < x - Px, Px - y >$. If possible let $\operatorname{Re} < x - Px, Px - y >< 0$. Let $\lambda \in (0, 1)$ and $u = \lambda y + (1 - \lambda) Px$. Note that $u \in C$. Consider $\|x - u\|^2 = \|x - \lambda y - (1 - \lambda) Px\|^2 =$

$< x - \lambda y - (1 - \lambda) Px, x - \lambda y - (1 - \lambda) Px >=< x - Px - \lambda(y - Px), x - Px - \lambda(y - Px) >$

$= \|x - Px\|^2 + \lambda^2 \|y - Px\|^2 + 2 \operatorname{Re} < x - Px, \lambda(Px - y) >$. For λ sufficiently small this last expression is less than $\|x - Px\|^2$ and this contradicts the definition of Px.

Remark: The definition only says $\|x-y\|^2 \geq \|x-Px\|^2 \; \forall y \in C$. It is interesting to note that there is always a lower bound for the difference $\|x-y\|^2 - \|x-Px\|^2$.

Problem 116

Let $\{x \in \mathbb{R}^n : \|x\| = 1\} \subset \bigcup_{j=1}^{n} \bar{B}(x_j, r_j)$ where $\bar{B}(x_j, r_j)$ is the closed ball with center x_j and radius r_j. Show that $0 \in \bar{B}(x_j, r_j)$ for some j. Show that the conclusion is false if the number of closed balls is allowed to exceed n.

For the counter example take $n = 2$ and consider the closed balls with centers at $2, -2, 2i, -2i$ and radius $\frac{3}{2}$ each. For the first part assume that $0 \notin \bar{B}(x_1, r_1)$. There is an $(n-1)-$ dimensional subspace M_{n-1} which is disjoint from $\bar{B}(x_j, r_j)$. We have $\{x \in M_{n-1} : \|x\| = 1\} \subset \bigcup_{j=2}^{n} \bar{B}(x_j, r_j)$. If $0 \notin \bar{B}(x_1, r_1)$ there is an $(n-2)-$ dimensional subspace M_{n-2} of M_{n-1} disjoint from $\bar{B}(x_2, r_2)$ and so on. If none of the given closed balls contains 0 then can repeat this argument until we get a $1-$ dimensional subspace M_1 such that $\{x \in M_1 : \|x\| = 1\} \subset \bar{B}(x_n, r_n)$. However $\bar{B}(x_n, r_n)$ is a convex set and if $x \in M_1$ with $\|x\| = 1$ then this ball contains both x and $-x$; hence it conatins $0 = \frac{x+(-x)}{2}$.

Problem 117

Let C be a closed convex set in a Hilbert space H. Let $P(x)$ be the point of C closest to x. Show that $\|P(x) - P(y)\| \leq \|x - y\| \; \forall x, y \in H$. [See also Problem 118 below].

If $y \in C$ we claim that $Q(x) = P(x)$ where $Q(x)$ is the point on the line segment $[y, P(x)]$ that is closest to x. (Since the line segment is a closed convex set $Q(x)$ exists). Assuming this claim we complete the proof as follows: let $x_1, x_2 \in H$. Apply the claim with $x = x_2, y = P(x_2)$ to conclude that $Q_1(x_2) = P(x_2)$ where Q_1 corresponds to the closed convex set $[P(x_2), P(x_1)]$. By symmetry $Q_1(x_1) = P(x_1)$. [We note that $[P(x_2), P(x_1)] = [P(x_1), P(x_2)]$!]. Now $\operatorname{Re} < x_1 - P(x_1), P(x_2) - P(x_1) > \leq 0$ and $\operatorname{Re} < x_2 - P(x_2), P(x_1) - P(x_2) > \leq 0$. [This was proved in Problem 115 above]. Rewrite the second inequality as $\operatorname{Re} < P(x_2) - x_2, P(x_2) - P(x_1) > \leq 0$ and add these two inequalities to get $\operatorname{Re} < x_1 - x_2 + P(x_2) - P(x_1), P(x_2) - P(x_1) > \leq 0$. Thus $\operatorname{Re} < x_1 - x_2, P(x_2) - P(x_1) > + \|P(x_2) - P(x_1)\|^2 \leq 0$. Finally this gives $\|P(x_2) - P(x_1)\|^2 \leq -\operatorname{Re} < x_1 - x_2, P(x_2) - P(x_1) > \leq \|x_2 - x_1\| \|P(x_2) - P(x_1)\|$ completing the proof. We now prove the claim. If the claim is false there exist $\lambda \in [0, 1]$ such that $\|x - \{\lambda y + (1-\lambda) P(x)\}\| < \|x - P(x)\|$. This can be written as $\left\|x + \frac{1-\lambda}{\lambda}\{x - P(x)\} - y\right\| <$

$\left\| \frac{1}{\lambda}x - \frac{1}{\lambda}P(x) \right\| = \left\| x + \frac{1-\lambda}{\lambda}\{x - P(x)\} - P(x) \right\|$. This would be a contradiction if we knew that $P(x)$ and $P(x + \frac{1-\lambda}{\lambda}\{x - P(x)\})$ were equal. To complete the proof we prove this last fact: (geometric meaning: the projection of x on C is also the projection on C of any point on the ray from x in the directioon of $x - P(x)$). Suppose this is false. There exists $y \in C$ such that $\left\| x + \frac{1-\lambda}{\lambda}\{x - P(x)\} - y \right\| < \left\| x + \frac{1-\lambda}{\lambda}\{x - P(x)\} - P(x) \right\|$. This gives $\|x - (1-\lambda)P(x) - \lambda y\| < \|x - (1-\lambda)P(x) - \lambda P(x)\| = \|x - P(x)\|$. This is clearly a contradiction.

Problem 118

In Problem 117 show that $\|P(x) - P(y)\| < \|x - y\|$ unless $P(x) - P(y) = x - y$.

Suppose $\|P(x) - P(y)\| = \|x - y\|$. From the argument used in Problem 117 we get $\operatorname{Re} < x-y+P(y)-P(x), P(y)-P(x) >= 0$. This gives $\|P(x) - P(y)\|^2 = 2\operatorname{Re} < x-y, P(x)-P(y) >$. Hence $\|\{P(x) - P(y)\} - (x-y)\|^2 = 2\|x-y\|^2 - 2\operatorname{Re} < P(x)-P(y), x-y >= 2\|x-y\|^2 - 2\|x-y\|^2 = 0$. Thus $P(x) - P(y) = x - y$.

Problem 119

Let $f : [0,\infty) \to [0,\infty)$ be non-decreasing with $\int\limits_1^\infty \frac{1}{f(x)}dx = \infty$. Show that $\int\limits_1^\infty \frac{1}{x\log(f(x))}dx = \infty$. Can we also assert that $\int\limits_1^\infty \frac{1}{x\log(f(x))\log(\log(f(x)))}dx = \infty$?

If $\int\limits_1^\infty \frac{1}{x\log(f(x))}dx < \infty$ we get $\int\limits_0^\infty \frac{1}{\log(f(e^y))}dy < \infty$. Hence $\frac{t/2}{\log(f(e^t))} \leq \int\limits_{t/2}^t \frac{1}{\log(f(e^y))}dy \to 0$ as $t \to \infty$. This gives $\frac{t/2}{\log(f(e^t))} < \frac{1}{4}$ and hence $\frac{e^t}{f(e^t)} < e^{-t}$ for large t. But then $\int\limits_1^\infty \frac{e^t}{f(e^t)}dt < \infty$ which means $\int\limits_1^\infty \frac{1}{f(x)}dx < \infty$, a contradiction. We now show that $\int\limits_1^\infty \frac{1}{x\log(f(x))\log(\log(f(x)))}dx$ can be finite. Let $k_n = e^{(e^{(e^n)})}, n = 0,1,2..$ and $f(x) = k_n$ on $[k_{n-1}, k_n)$ for all $n \geq 1$. We see that $\int\limits_{e^e}^\infty \frac{1}{x\log(f(x))\log(\log(f(x)))}dx < \sum e^{-n} < \infty$.

Problem 120

a) Let (X, d) be a compact metric space and $T : X \to X$ be onto. If $d(Tx, Ty) \leq d(x, y)$ $\forall x, y$ prove that $d(Tx, Ty) = d(x, y)$ $\forall x, y$.

b) Let (X, d) be a compact metric space and a continuous map $T : X \to X$ satisfy $d(Tx, Ty) \geq d(x, y)$ $\forall x, y$. Prove that the conclusion of part a) holds.

Remark: several improvements of these results are given in the next few problems. See also Problem 423.

For each n, T^n (the $n-th$ iterate of T) is onto. Given x, y we can find x_n and y_n such that $T^n(x_n) = x$ and $T^n(y_n) = y$. By compactness we can find $n_1 < n_2 < ...$ such that $\{x_{k_k}\}$ converges to some element u and $\{y_{k_k}\}$ converges to some element v. Note that $d(T^n u, T^n v)$ is increasing in n. Now, $d(x, T^{n_k}(u)) = d(T^{n_k}(x_{n_k}), T^{n_k}(u)) \leq d(x_{n_k}, u) \to 0$. Thus $T^{n_k}(u) \to x$ as $k \to \infty$. Similarly $T^{n_k}(v) \to y$ as $k \to \infty$. It follows that $d(T^{n_k}(u), T^{n_k}(v)) \to d(x, y)$. Monotonicity of the sequence $\{d(T^n u, T^n v)\}$ shows that $d(T^n(u), T^n(v)) \to d(x, y)$. In particular $d(T(T^{n_k}(u)), T(T^{n_k}(v))) \to d(x, y)$. Therefore $d(x, y) = \lim d(T(T^{n_k}(u)), T(T^{n_k}(v))) = d(Tx, Ty)$ because $T^{n_k}(u) \to x$ as $k \to \infty$ and $T^{n_k}(v) \to y$ and T is continuous.

b) Claim: $T(X) = X$. If not there is an element y in $X \backslash T(X)$. Since $T(X)$ is compact, $\alpha \equiv d(y, T(X)) > 0$. Now $\alpha \leq d(y, T^k(y) \leq d(T^n y, T^{n+k} y)$ for all positive integers n and k. It follows that $\{T^n y\}$ has no convergent subsequence contradicting compactness of X. Thus T is onto and we can apply part a) to T^{-1} to complete the proof.

Problem 121

Let (X, d) be a compact metric space and $T : X \to X$ satisfy $d(Tx, Ty) \geq d(x, y)$ for all $x, y \in X$. Then T is an isometry of X onto itself. [Thus continuity of T need not be assumed in previous problem]

[See also Problem 234 below]

Let $x_0, y_0 \in X$ and $\delta > 0$. Since X can be covered by a finite number of open balls of any given radius there is an open ball of radius $\delta/4$ containing infinitely many of the points $T^n x_0$ and an open ball of radius $\delta/4$ containing infinitely many of the points $T^n y_0$. Let these balls be $B(u, \delta/4)$ and $B(v, \delta/4)$. Let $n_1 < n_2 < ...$ with $T^{n_k} x_0 \in B(u, \delta/4)$ and $T^{n_k} y_0 \in B(v, \delta/4)$ $\forall k \geq 1$. Let $k \leq l$. Then $d(T^{n_k} x_0, T^{n_l} x_0) \leq d(T^{n_k} x_0, u) + d(T^{n_l} x_0, u) < \delta/2$ and (similarly) $d(T^{n_k} y_0, T^{n_l} y_0) < \delta/2$. Hence $d(x_0, T^{n_l - n_k} x_0) \leq d(T^{n_k} x_0, T^{n_k} T^{n_l - n_k} x_0) < \delta/2$ and (similarly) $d(y_0, T^{n_l - n_k} y_0) < \delta/2$. Thus, $d(Tx_0, Ty_0) \leq d(T^{n_l - n_k} x_0, T^{n_l - n_k} y_0) \leq \delta/2 + d(x_0, y_0) + \delta/2$. Since $x_0, y_0 \in X$ and $\delta > 0$ are arbitrary we see that T is an isometry. It remains to show that T is onto. The sets $X, T(X), T^2(X), ...$ are all compact and this sequence is decreasing. The sequence has finite intersection property and hence $X_\infty = \bigcap_{n=0}^{\infty} T^n(X)$ is non-empty. Note that T maps X_∞ into itself. In fact, $T(X_\infty) = X_\infty$. [Since T is an isometry it is one-to-one]. Let $X_\tau = \{x \in X : d(x, X_\infty) \geq \tau\}$ for each $\tau > 0$. X_τ is closed. Suppose $X_\tau \neq \emptyset$.

Then $X_\tau, T(X_\tau), T^2(X_\tau), \ldots$ is a decreasing sequence of compact sets with finite intersection property. Hence there is a point w in the intersection $X_{\tau,\infty}$ of these sets. But $X_{\tau,\infty} \subset X_\tau \cap X_\infty$. This is a contradiction because $w \in X_\tau$ and so $d(w, X_\infty) \geq \tau$ whereas $d(w, X_\infty) = 0$. This proves that $X_\tau = \emptyset$ for every $\tau > 0$ which means $d(x, X_\infty) = 0$ for all $x \in X$. Thus $X = X_\infty \subset T(X)$ and T is onto.

Problem 122

Find an error in the following proof given in Amercan Math. Monthly, vol. 98, no. 7, 1991 (p. 664).

Let X be a compact metric space and $T : X \to X$ be any map with $\inf\limits_{n \geq 1} d(T^n x, T^n y) > 0$ whenever $x \neq y$. Show that $T(X) = X$. Solution: let $D(x,y) = \inf\limits_{n \geq 0} d(T^n x, T^n y)$ where $T^0 = I$. D is a metric and $D \leq d$. It follows by compactness of X that the identity map $i : (X, d) \to (X, D)$ is a homeomorphism and (X, D) is a compact metric space. By definition $D(Tx, Ty) \geq D(x, y)$. By Problem 121 above T is an isometry of X onto itself.

Why is D a metric? Minimum of two metrics need not be a metric. Example: $X = \{0, 1, 2\}$, d the usual metric and $D(0,1) = .5, D(1,2) = 1.5, D(0,2) = 2$. Note that if $d' = \min\{d, D\}$ then $d'(0,1) = .5, d'(0,2) = 2$ and $d'(1,2) = 1$ so $d'(0,2) > d'(0,1) + d'(1,2)$. [It is not clear if the statement above is true].

Problem 123

Is the product of two derivatives on $\mathbb{R}$ necessarily a derivative?

No! Let $\phi(x) = x \sin(\frac{1}{x})$ if $x \neq 0$ and $\phi(0) = 0$. Then $(x\phi(x))' = x\phi'(x) + \phi(x) = \phi(x) - \phi_1(x) + \phi(x)$ where $\phi_1(x) = \cos(\frac{1}{x})$ if $x \neq 0$ and $\phi_1(0) = 0$. It follows that $2\phi(x) - \phi_1(x) = (x\phi(x))'$. Being continuous ϕ is a derivative and so is ϕ_1 (because $2\phi(x) - \phi_1(x)$ is a derivative). Let $f' = \phi_1$. We claim that ϕ_1^2 is not a derivatine: suppose $g' = \phi_1^2$. Then $(g(x) - f(\frac{x}{2}))' = \phi_1^2(x) - \frac{1}{2}\phi_1(x/2) = \cos^2(\frac{1}{x}) - \frac{1}{2}\cos(\frac{2}{x}) = \frac{1}{2}$ if $x \neq 0$ and 0 if $x = 0$. This is absurd: $g(x) - f(\frac{x}{2})$ has to be of the type $x + a$ on $(0, \infty)$ and $x + b$ on $(-\infty, 0)$ and $a = b$ by continuity at 0; but thenthe derivative at 0 is 1, not 0!

Problem124

Let $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ and f, g be non-negative continuous functions on $\mathbb{R}$ with compact support. Show that $\int \sup\limits_y \{f(x-y)g(y)\} dx \geq \|f\|_p \|g\|_q$.

If $\|f\|_\infty = f(a)$ then $\sup\limits_y \{f(x-y)g(y)\} \geq f(a)g(x-a) = \|f\|_\infty g(x-a)$ and hence $\int \sup\limits_y \{f(x-y)g(y)\} dx \geq \|f\|_\infty \|g\|_1$. Similarly if $g(b) = \|g\|_\infty$ then $\sup\limits_y \{f(x-y)g(y)\} \geq g(b)f(x-b) = \|g\|_\infty f(x-b)$ so $\int \sup\limits_y \{f(x-y)g(y)\} dx \geq$

$\|g\|_\infty \|f\|_1$. Thus $\int \sup_y \{f(x-y)g(y)\}dx \geq \max\{\|f\|_\infty \|g\|_1, \|g\|_\infty \|f\|_1\}$. Hence $\int \sup_y \{f(x-y)g(y)\}dx \geq \|f\|_\infty^{1/q} \|g\|_1^{1/q} \|g\|_\infty^{1/p} \|f\|_1^{1/p}$. Now $\|g\|_1^{1/q} \|g\|_\infty^{1/p} = (\|g\|_\infty^{q-1} \|g\|_1)^{1/q} \geq (\int g^q)^{1/q} = \|g\|_q$. Similarly $\|f\|_\infty^{1/q} \|f\|_1^{1/p} \geq \|f\|_p$ and the proof is complete.

Problem 125

a) Find all positive numbers α such that there is a positive C^1 function f on $(0,\infty)$ with $f'(x) \geq a[f(x)]^\alpha$ for all x sufficiently large for some $a \in (0,\infty)$.

b) Does there exist a positive C^1 function f on $(0,\infty)$ with $f'(x) \geq af(f(x))$ for all x sufficiently large for some $a \in (0,\infty)$?

a) For $\alpha \leq 1$ such a function exists: take $f(x) = e^x$. Let $\alpha > 1$. There is a positive integer n such that $n < \alpha \leq n+1$. We have $\frac{f'(x)}{[f(x)]^n} \geq a[f(x)]^{\alpha-n}$ for $x \geq T$ (say) and so $\frac{[f(x)]^{1-n}}{1-n} \geq \frac{[f(T)]^{1-n}}{1-n} + a\int_T^x [f(t)]^{\alpha-n}dt$ for $x > T$. Note that $f'(x) > 0$ and so f is increasing on (T,∞). Thus $f(x) \geq f(T)$ and $\int_T^x [f(t)]^{\alpha-n}dt \geq [f(T)]^{\alpha-n}(x-T)$. Finallt this gives $\frac{[f(x)]^{1-n}}{1-n} \geq \frac{[f(T)]^{1-n}}{1-n} + a[f(T)]^{\alpha-n}(x-T)$. In other words $\{\frac{[f(T)]^{1-n}}{1-n} + a[f(T)]^{\alpha-n}(x-T)\}[f(x)]^{n-1}$ is bounded. This is a contradiction because $f'(x) \geq a[f(T)]^\alpha$ which implies (by Mean Value Theorem) that $f(x) \to \infty$ as $x \to \infty$.

b) The answer is no. As in a) we get $f'(x) \geq af(f(T))$ for $x \geq T$ which implies $f(x) \to \infty$ and hence $f'(x) \to \infty$. It follows [by Mean Value Theorem] that $f(x) > x + \frac{1}{a}$ for some x. By Mean Value Theorem applied to $[x, f(x)]$ we get $f(f(x)) - f(x) = f'(\beta)[f(x) - x]$ for some $\beta \in (x, f(x))$. Thus $f(f(x)) \geq f(x) + af(f(x))[f(x) - x]$. This says $f(x) + f(f(x))\{a[f(x) - x] - 1\} \leq 0$ which is absurd.

Problem 126

Let $a_n > 0$ and $\sum_{n=1}^\infty a_n \log(1 + \frac{1}{a_n}) < \infty$. Show that $\sum_{n=1}^\infty \frac{a_n}{\|x-b_n\|^k} < \infty$ almost everywhere for any sequence $\{b_n\} \subset \mathbb{R}^k$. [$\|\|$ is the norm in $\mathbb{R}^k$].

Using the fact that $a_n \log(1 + \frac{1}{a_n}) \to 0$ we see that 0 is the only possible limit point of $\{a_n\}$ in $[0,\infty]$. Thus $a_n \to 0$. Since $\log(1 + \frac{1}{a_n}) \to \infty$ it follows that $\sum_{n=1}^\infty a_n < \infty$. Let $0 < R < \infty$. We prove that $\sum_{n=1}^\infty \frac{a_n}{\|x-b_n\|^k} < \infty$ almost everywhere on $B_R \equiv \{x \in \mathbb{R}^k : \|x\| \leq R\}$. This would finish the proof since R is arbitrary. By deleting the first few terms of the series $\sum_{n=1}^{\infty} \frac{1}{\|x-b_n\|^k}$ we may assume that $a_n \leq 2R\ \forall n$. Also we can split the sum $\sum_{n=1}^{\infty} \frac{a_n}{\|x-b_n\|^k}$ into two parts: the one in which $\|b_n\| \leq 2R$ and the one in which $\|b_n\| > 2R$. In the second sum we have $\|x-b_n\|^k > (2R-R)^k$ and $\frac{a_n}{\|x-b_n\|^k} < \frac{a_n}{R^k}$. Since $\sum_{n=1}^{\infty} a_n < \infty$ the second sum is finite. In the first sum $\|b_n\| \leq 2R$ for all n. We now define $Y_n = \frac{1}{\|X-b_n\|^k}$ if $\|x-b_n\|^k \geq a_n$ and $Y_n = 0$ otherwise where X is uniformly distributed over the ball B_R. If we prove that $\sum a_n Y_n < \infty$ a.s. we are done because $Y_n = \frac{1}{\|X-b_n\|^k}$ eventually w.p. 1. [We use Borel-Cantelli Lemma to justify this. We have to show that $\sum P\{\|X-b_n\|^k < a_n\} < \infty$. But $P\{\|X-b_n\|^k < a_n\} \leq P\{X^{-1}(B(b_n, a_n^{1/k}))\} \leq \frac{m_k(B(b_n, a_n^{1/k}))}{m_k(B_R)}$ where m_k is $k-$ dimensional Lebesgue measure. Thus $P\{\|X-b_n\|^k < a_n\} \leq \frac{a_n}{R^k}$]. Now $EY_n = \int\limits_{a_n \leq \|x-b_n\|^k} \frac{1}{\|x-b_n\|^k} dm_k / cR^k$ (where c is the volume of the unit ball in $\mathbb{R}^k$. Noting that $\|x-b_n\| \leq 3R$ we see that $EY_n \leq \int\limits_{a_n \leq \|u\|^k \leq (3R)^k} \frac{1}{\|u\|^k} dm_k(u)/cR^k$. Using spherical coordinates in $\mathbb{R}^k$ we see that $EY_n \leq C \int\limits_{a_n^{1/k}}^{3R} \frac{1}{t^k} t^{k-1} dt = C[\log(3R) - \log(a_n^{1/k})]$ for some constant C. It remains only to see that $\sum_{n=1}^{\infty} a_n \log \frac{3R}{a_n^{1/k}} < \infty$. Since $\log \frac{3R}{a_n^{1/k}} = \log(3R) + \frac{1}{k}\log(\frac{1}{a_n}) < \log(3R) + \frac{1}{k}\log(1 + \frac{1}{a_n})$ the proof is complete.

Problem 127

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f \circ g$ is Riemann integrable on $[0,1]$ whenever $g : [0,1] \to \mathbb{R}$ is continuous. Show that f is continuous on $\mathbb{R}$.

Let C be a Cantor-like set of positive measure in $[0,1]$. Let $h : [0,1] \to \mathbb{R}$ be defined by $h(x) = d(x, C)\ \forall x \in \mathbb{R}$. Let $a \in \mathbb{R}$ and $g(x) = a + h(x)$. By hypothesis $f \circ g$ is Riemann integrable on $[0,1]$. Hence it is continuous a.e. In particular there is a point $c \in C \backslash \{0,1\}$ such that $f \circ g$ is continuous at c. Let $\epsilon > 0$ and choose $\delta > 0$ such that $|(f \circ g)(x) - (f \circ g)(c)| < \epsilon$ if $|x - c| < \delta$ (and $c + \delta < 1$). Let $d \in (c, c+\delta) \backslash C$ and $y \in [g(c), g(d)] \equiv [a, a + h(d)]$. Note that such a d exists because $C^0 = \emptyset$; also $h(d) > 0$. Since g is continuous, g must attain the value y at some point x between c and d. Thus $c < x < d$ and $g(x) = y$. Now $|f(y) - f(a)| = |f(y) - f(a + h(c))| = |f(g(x)) - f(g(c))| < \epsilon$ since $|x - c| < d - c < \delta$. We have proved that given any real number a there is a right-hand interval $[a, a + h(d)]$ on which $|f(.) - f(a)| < \epsilon$. This proves right continuity of f. Applying this to $f(-x)$ we see that f is also left-continuous at each point.

Problem 128

Is the set of all $n \times n$ invertible matrices dense in the space of all $n \times n$ matrices? Is the space of all invertible operators on a Hilbert space dense in the space of all operators on that space?

If A is any $n \times n$ matrix then we can find $\delta_n \downarrow 0$ such that $A + \delta_n I$ is invertible for each n. Hence the answer to the first question is 'yes'. The answer to the second question is 'no'. Let $H = l^2$ and $T\{x_n\} = \{0, x_1, x_2, ...\}$. We claim that no operator S on H satisfying $\|T - S\| < 1$ is invertible. Indeed, if we define T_1 by $T_1(\{x_1, x_2, ...\}) = \{x_2, x_3, ..\}$. Then $T_1 T = I$ so $\|I - T_1 S\| = \|T_1 T - T_1 S\| \leq \|T - S\| < 1$ implying that $T_1 S$ is invertible. This implies that T_1 itself is invertible which is obviously false.

Problem 129

Let A be any $n \times n$ matrix. For any positive integer k Show that there is a unique $n \times n$ matrix B such that $B(B^*B)^k = A$.

Existence is an easy consequence of the fact that we can factor A as UP where U is unitary and P is non-negative definite. [See e.g. Linear Algebra by Hoffman and Kunze, p. 342. U is not unique in general, is it is interesting that B is unique]. We define B as UQ where $(2k+1) - th$ root of P. Then Q is non-negative definite and $Q^{2k+1} = P$. Thus $B(B^*B)^k = UQ(Q^*U^*UQ)^k = UQ^{2k+1} = UP = A$. This proves existence. Suppose $B(B^*B)^k = A$ and $C(C^*C)^k = A$. Then $A^*A = (B^*B)^k B^*B(B^*B)^k = (B^*B)^{2k+1}$. Also $A^*A = (C^*C)^{2k+1}$. It follows that $B^*B = C^*C$. Now we see easily that $\ker(B) = \ker(C) = \ker(B^*B) = \ker(C^*C) = M$ (say). Thus B and C agree on M. They also agree at any eigen vector of $B^*B = C^*C$ corresponding to a non-zero eigem value: $B^*Bx = \lambda x, \lambda \neq 0, x \neq 0$ implies $Ax = B(B^*B)^k x = B\lambda^k x = \lambda^k x$ and, similarly, $Ax = \lambda^k Cx$ so $Bx = Cx$. It follows now that B and C agree everywhere.

Probem 130

Let $f : \mathbb{R} \to \mathbb{R}$. Then f is continuous at 0 if and only if $f(x_n) \to 0$ wheneve $x_n \to 0$. Can differentiability of f be characterized by the condition $\sum f(x_n)$ converges whenever $\sum x_n$ converges?

We prove that if the stated condition holds then f is differentiable at 0. The example $f(x) = x^2, x_n = \frac{(-1)^n}{\sqrt{n}}$ show that the converse is false. Problem 131 below gives more information on this]. Let us say that f is CP (convergence preserving) if $\sum f(x_n)$ converges whenever $\sum x_n$ converges. Note that if f and g are CP then so are $af+bg$ and $f \circ g$ (for $a, b \in \mathbb{R}$). Claim : f CP implies there exists $M \in (0, \infty)$ and $\epsilon > 0$ such that $f(x) < Mx$ whenever $0 < x < \epsilon$. If this is false we can find $x_n's$ such that $0 < x_n < \frac{1}{n^2}$ and $f(x_n) \geq nx_n \ \forall n$. Let k_n be the least positive integer $\geq \frac{1}{n^2 x_n}$. Consider the series $x_1 + x_1 + ... + x_1 (k_1$ terms$) + x_2 + x_2 + ... + x_2 (k_2$ terms$) + ...$ The series converges because $k_n x_n < (1 + \frac{1}{n^2 x_n}) x_n < \frac{2}{n^2}$. The series $f(x_1) + f(x_1) + ... + f(x_1)(k_1$ terms$) + f(x_2) + f(x_2) + ... + f(x_2)(k_2$ terms) does not converge because $k_n f(x_n) \geq n x_n k_n \geq \frac{1}{n}$. This proves the claim. To prove that f is differentiable at 0 we first show that $D_- f(0) \geq D^+ f(0)$. [In this notation $+$ and $-$ signs stand for limits from the right and left and subscript/superscript stand for limit inferior and limit superior. Thus $D_- f(0) = \liminf\limits_{h \uparrow 0} \frac{f(h) - f(0)}{h}$. Note that by taking $x_n = 0$ for all n in the hypothesis we get $f(0) = 0$]. If possible let $D_- f(x) < D^+ f(0)$. Let $D_- f(x) < s < t < D^+ f(0)$. Let $c_n \downarrow 0, 0 < c_n < \frac{1}{2^n}$ and $\frac{f(c_n)}{c_n} > t$ for all n. Let $d_n \uparrow 0, -\frac{1}{2^n} < d_n < 0$ and $\frac{f(d_n)}{d_n} < s$ for all n. We may suppose $c_1 > -d_1 > c_2 > -d_2.....$ Let k_n be the least integer with $k_n c_n > \frac{1}{n}$ and l_n the smallest integer with $l_n d_n < -\frac{1}{n}$. Consider the series $\sum x_n$ where the first k_1 terms are c_1, the next l_1 are d_1, the next k_2 are c_2, the next l_2 are d_2 and so on. $\sum f(x_n)$ is not convergent because $t k_n c_n + s l_n d_n = (t - s) k_n c_n + s(k_n c_n + l_n d_n), \sum k_n c_n$ diverges and $\sum (k_n c_n + l_n d_n)$ converges. Since $\sum x_n$ converges and $\sum f(x_n)$ diverges we have proved that $D_- f(0) \geq D^+ f(0)$. Replacing $f(x)$ by $f(-x)$ we get $D_+ f(0) \geq D^- f(0)$. Hence $D^+ f(0) \leq D_- f(0) \leq D^- f(0) \leq D_+ f(0) \leq D^+ f(0)$ proving that $f'(0)$ exists. [Note that the claim above proves that $D^+ f(0) < \infty$. Hence $f'(0) < \infty$. Changing f to $-f$ we see that $f'(0) > -\infty$].

Problem 131

Find a necessary and sufficient condition for f to be CP. [See Problem 130 for definition of CP].

The condition is $f(x) = \alpha x$ for all x in some neighbourhood of 0.

By considering $f(x) - f'(0)x$ we may reduce the proof to the case $f'(0) = 0$. We have to show that $f \equiv 0$ in a neighbourhood of 0. Claim: given $A, B > 0$ and $\epsilon > 0$ we can find a positive integer k and $\delta > 0$ such that $\delta < \epsilon, A < k\delta < A + \epsilon$ and $k |f(\delta)| < B$. We can also find a positive integer k and $\delta > 0$ such that $-\delta < \epsilon, A < -k\delta < A + \epsilon$ and $k |f(-\delta)| < B$. To see this assume $\epsilon < 1$ and choose δ such that $\frac{|f(\delta)|}{\delta} < \frac{B}{A+1}$ and $\delta < \epsilon$. The interval $(\frac{A}{\delta}, \frac{A+\epsilon}{\delta})$ contains an integer k and $k |f(\delta)| < \frac{A+\epsilon}{\delta} |f(\delta)| < B$. For the second oart pick k in $(-\frac{A+\epsilon}{\delta}, -\frac{A}{\delta})$ where $\frac{|f(-\delta)|}{\delta} < \frac{B}{A+1}$. This proves the claim. To show that $f \equiv 0$ in a neighbourhood of 0, suppose this is false so that we can find $y_n's$ with $0 < f(y_n) < y_n < \frac{1}{2^n}$. [in order that y_n and $f(y_n)$ can both be takeb to be positive we may have tp replace $f(x)$ by $f(-x), -f(x)$ or $-f(-x)$]. Let m_n be the least integer with $m_n f(y_n) > \frac{1}{n}$. Let δ_n, k_n be such that $\delta_n < \frac{1}{2^n}, m_n f(y_n) < -k_n\delta_n < m_n f(y_n) + \frac{1}{2^n}$ and $k_n |f(-\delta_n)| < \frac{1}{n^2}$. Look at the series whose terms are $y_1, y_1, ..., y_1(m_1$ times$), -\delta_1, -\delta_1, ..., -\delta_1(k_1$ times$), y_2, y_2, ..., y_2(m_2$ times$), -\delta_2, -\delta_2, ..., -\delta_2(k_2$ times$), ...$ This series converges but the the series whose terms are images under f of the terms of this series diverges.

Problem 132

Let $f : (0,1) \to (0,1)$ be a continuous function such that for any $x \in (0,1)$ there is an integer n such that $f_{(n)}(x) = x$ where $f_{(1)} = f$ and $f_{(n)} = f \circ f_{(n-1)}$ for $n \geq 2$. Show that $f(x) = x \; \forall x \in (0,1)$. Is the result true of $(0,1)$ is replaced by $[0,1]$?

Clearly, f is onto. If $f(x) = f(y)$ then choose n, m with $f_{(n)}(x) = x$ and $f_{(m)}(y) = y$. We have $f_{(nm)}(x) = x$ and $f_{(nm)}(y) = y$ and this implies $x = y$. Thus f is a continuous bijection of $(0,1)$. Hence f is strictly monotonic. If it is increasing and $f(x) > x$ for some x then (for suitable n) $x = f_{(n)}(x) > x$ a contradiction. Similarly if $f(x) < x$ then $x = f_{(n)}(x) < x$ a contradiction again. Thus, if f is increasing then it must be the identity. Now assume that f is strictly decreasing. Suppose $f(f(x)) > x$ and $f_{(n)}(x) = x$. Then $f_{(2n)}(x) > x$ by monotonicity and iteration of $f(f(x)) > x$. This contradicts the fact that $f_{(2n)}(x) = x$. Similarly $f(f(x)) < x$ leads to a contradiction. Hence $f \circ f = f$. We have $f(x) > x$ implies $f(f(x)) < f(x)$ and $f(x) < x$ implies $f(f(x)) > f(x) = 0$ both of which contradict $f \circ f = f$. Thus f is the identity function. We cannot draw the same conclusion when $(0,1)$ is replaced by $[0,1]$. For example $f : [0,1] \to [0,1]$ defined by $f(x) = 1 - x$ shows that f need not be the identity. However, the proof above does show that $f \circ f = f$ in this case also.

Problem 133

a) Let $f : \{0,1,2...\} \to \{0,1,2...\}$ satisfy $f(m^2 + n^2) = f^2(m) + f^2(n)$ $\forall m, n \geq 0$. Show that either $f(n) = 0$ for all n or $f(n) = n$ for all n

b) Let $f : [0,\infty) \to [0,\infty)$ satisfy $f(x^2 + y^2) = f^2(x) + f^2(y) \; \forall x, y \geq 0$. If f is continuous show that $f \equiv 0$ or $f \equiv \frac{1}{2}$ or $f(x) = x$ for all x.

a) We have $f(0) = 2f^2(0)$. Since $f(0)$ is an integer we get $f(0) = 0$. Next $f(1) = f^2(1) + f^2(0)$ so $f(1) = 0$ or 1. If $f(1) = 1$ we prove that $f(n) = n$ for all. A similar argument shows that if $f(1) = 0$ then $f(n) = 0$ for all n. So let $f(1) = 1$. Then $f(2) = f^2(1) + f^2(1) = 2$. Also $f(5) = f^2(2) + f^2(1) = 5$ and $f(4) = f^2(2) + f^2(0) = 4$. Since $3^2 + 4^2 = 5^2$ we get $f^2(3) + f^2(4) = f^2(5) + f^2(0)$ and this gives $f^2(3) = 25 - 16 = 9$ so $f(3) = 3$. $7^2 + 1^2 = 5^2 + 5^2$ we get $f^2(7) + f^2(1) = f^2(5) + f^2(5)$ and this gives $f^2(7) = 49$ and $f(7) = 7$. Clearly

$f(9) = f(3^2) + f(0^2) = f^2(3) + 0 = 9$. $f(10) = f(3^2 + 1^2) = f^2(3) + f^2(1) = 10$ and $f(8) = f(2^2 + 2^2) = f^2(2) + f^2(2) = 8$. The equation $6^2 + 8^2 = 10^2$ gives $f(6) = 6$. Thus $f(n) = n$ for $n \leq 10$. The identity $(2k+1)^2 + (k-2)^2 = (2k-1)^2 + (k+2)^2$ gives $f(2k+1) = 2k+1$ if we assume that $f(m) = m$ for $m < n$. Similarly $(2k+2)^2 + (k-4)^2 = (2k-2)^2 + (k+4)^2$ gives $f(2k+2) = 2k+2$ provided $k > 2$ if we assume that $f(m) = m$ for $m < n$. We have proved that $f(n) = n$ for all n.

b) We have $f(0) = 2f^2(0)$ so $f(0) = 0$ or $f(0) = \frac{1}{2}$. First let $f(0) = 0$. Since $f(1) = f^2(1) + f^2(0)$ we get $f(1) = 0$ or 1. We also have $f(x^2) = f(x^2 + 0^2) = f^2(x) + 0$ so $f(x^2 + y^2) = f(x^2) + f(y^2)$ $\forall x, y \geq 0$. This implies that f is additive; a simple argument shows $f(rx) = rf(x)$ for every positive rational r and continuity yields $f(xy) = yf(x)$ $\forall x, y \geq 0$. Putting $x = 1$ we get $f(y) = y$ $\forall y \geq 0$ or $f(y) = 0$ $\forall y \geq 0$. Now let $f(0) = \frac{1}{2}$. Then $f(x^2) = f(x^2 + 0^2) = f^2(x) + \frac{1}{4}$ so $f(x^2 + y^2) = f(x^2) - \frac{1}{4} + f(y^2) - \frac{1}{4}$ $\forall x, y \geq 0$. This means $f(x^2 + y^2) - \frac{1}{2} = f(x^2) - \frac{1}{2} + f(y^2) - \frac{1}{2}$ $\forall x, y \geq 0$. As above it follows that the additive function $f(t) - \frac{1}{2}$ is of the type ct for some constant c and the only possibility is $c = 0$ so $f(x) = \frac{1}{2}$ for all x.

Problem 134

Let C be a bounded subset of $V \equiv \mathbb{R}^n$ or $\mathbb{C}^n$ such that for each $x \in V$ there is a unique point Px of C which is closest to it. Show that C is closed and convex.

It is trivial to see that C is closed: if $\{c_n\} \subset C$ and $c_n \to x$ then $\|x - Px\| \leq \|x - c_n\| \to 0$ so $x = Px \in C$. Convexity of C requires a lengthy argument and we divide the proof into the steps S1-S4 as follows:

S1. If $x \in V, \lambda \geq 0$ and $x_\lambda = x + \lambda(x - Px)$ then $Px_\lambda = Px$. (Geometrically x_λ is a point on the ray from x in the direction of $x - Px$. S1 says the point of C closest to any point on this ray is Px).

S2. If $x \in V$ and $y \in C$ then the line segment $[y, Px]$ is closed and convex and if Qz is the point of $[y, Px]$ closest to z (which exists for any $z \in V$ by a standard result in Functional Analysis) then $Px = Qx$.

S3.$\|Px - Py\| \leq \|x - y\|$ $\forall x, y \in V$.

S4. C is convex.

We now proceed backwards: suppose we have proved S1-S3. Suppose C is not convex. Since C is closed we can find $x, y \in C$ such that $u \equiv \frac{x+y}{2} \notin C$. We claim that either $\|x - Pu\| > \left\|\frac{x-y}{2}\right\|$ or $\|y - Pu\| > \left\|\frac{x-y}{2}\right\|$. In the contrary case $\|x - y\| \leq \|x - Pu\| + \|Pu - y\| \leq \left\|\frac{x-y}{2}\right\| + \left\|\frac{x-y}{2}\right\| = \|x - y\|$. This implies that $\|x - Pu\| = \left\|\frac{x-y}{2}\right\|, \|Pu - y\| = \left\|\frac{x-y}{2}\right\|$ and that $x - Pu = t(Pu - y)$ for some $t \geq 0$. Taking norms on both sides we get $t = 1$ so $Pu = u$ contradicting the fact that $u \notin C$. Suppose, for definiteness, that $\|x - Pu\| > \left\|\frac{x-y}{2}\right\|$. Since $x, y \in C$ we get $\|Px - Pu\| = \|x - Pu\| > \left\|\frac{x-y}{2}\right\| = \|x - u\|$ contradicting S3.

Next we prove S3 assuming that S1 and S2 hold. Let $Q_{[u,v]}$ be the projection onto the line segment $[u, v]$. (This means $Q_{[u,v]}z$ is the point of $[a, b]$ closest to

z for any $z \in V$). By S2 with y replaced by Py, $Px = Q_{[Py,Px]}x$. Similarly $Py = Q_{[Px,Py]}y$. Since the line segments $[Px, Py]$ and $[Py, Px]$ are identical this gives $Px = Qx$ and $Py = Qy$ where $Q = Q_{[Px,Py]}$. To prove S3 we only have to show that $\|Qx - Qy\| \leq \|x - y\| \ \forall x, y \in V$. The crucial point here is that Q is projection on a *convex* set. We note that for $0 < t < 1$, $\|x - Qx\|^2 \leq \|x - \{tQx + (1-t)Qy\}\|^2 = \|x - Qx\|^2 + (1-t)^2 d + 2(1-t) \operatorname{Re} < x - Qx, Qx - Qy >$. This gives $0 \leq (1-t)\|Qx - Qy\|^2 + 2 \operatorname{Re} < x - Qx, Qx - Qy >$. Letting $t \uparrow 1$ we get $\operatorname{Re} < x - Qx, Qx - Qy > \geq 0$. Interchanging x and y we get $\operatorname{Re} < y - Qy, Qy - Qx > \geq 0$. Equivalently, $\operatorname{Re} < Qy - y, Qx - Qy > \geq 0$. Adding thes two inequalities we get $\operatorname{Re} < x - y + Qy - Qx, Qx - Qy > \geq 0$. In other words, $\|Qx - Qy\|^2 \leq \operatorname{Re} < x - y, Qx - Qy > \leq \|x - y\| \|Qx - Qy\|$ and $\|Qx - Qy\| \leq \|x - y\|$.

Proof of S2 using S1: If S2 fails then $\exists\ x \in V, y \in C$ such that $Px \neq Qx$ where Q is the projection on $[y, Px]$. There exists $t \in (0,1)$ such that $\|x - x_t\| < \|x - Px\|$ where $x_t = ty + (1-t)Px$. Thus $\left\|x + \frac{1-t}{t}(x - Px) - y\right\| < \left\|\frac{1}{t}x - \frac{1}{t}Px\right\| = \left\|x + \frac{1-t}{t}(x - Px) - Px\right\|$. This is a contradiction.

Proof of S1.

Suppose $Px_\lambda \neq Px$ for some $\lambda \geq 0$ where $x_\lambda = x + \lambda(x - Px)$. We claim that $Px_t = Px$ for some $t \geq 0$ implies that $Px_s = Px$ for $0 \leq s < t$. To see this we write x_s as $(1-\tau)x_t + \tau Px$ where $\tau = \frac{t-s}{1+t}$. Note that $0 < \tau < 1$. If $Px_s \neq Px (= Px_t)$ then $\|x_s - Px\| > \|x_s - v\|$ for some $v \in C$. But $\|x_t - v\| \leq \|x_t - x_s\| + \|x_s - v\| < \|x_t - x_s\| + \|x_s - Px\| = \|x_t - (1-\tau)x_t - \tau Px\| + \|(1-\tau)x_t + \tau Px - Px\|$

$= \tau \|x_t - Px\| + (1-\tau)\|x_t - Px\| = \|x_t - Px\|$ which contradicts the fact that $Px_t = Px$.

Let $\alpha = \sup\{t \geq 0 : Px_t = Px\}$. If $\alpha = \infty$ then $Px_r = Px$ for all r but $Px_\lambda \neq Px$. Hence $\alpha < \infty$. Clearly $Px_s = Px$ for $s < \alpha$ and $Px_s \neq Px$ for $s > \alpha$. We now prove that $Px_\alpha = Px$. For $\lambda < \alpha$, $Px_\lambda = Px$. Hence, for any $w \in C$, $\|x_\lambda - Px\| \leq \|x_\lambda - w\|$. Letting $\lambda \uparrow \alpha$ we get $\|x_\alpha - Px\| \leq \|x_\alpha - w\|$. This proves that $Px_\lambda = Px$ if and only if $\lambda \leq \alpha$. Now let D be a closed ball with center x_α (say with radius r) which is disjoint from C. For $z \in D$ let $F(z) = x_\alpha + \frac{r}{\|x_\alpha - Pz\|}[x_\alpha - Pz]$. If we prove that P is continuous then (since $\|x_\alpha - Pz\| \geq r$), we see that F is continuous on D. Also, $\|x_\alpha - F(z)\| = r$ so F maps D into its boundary. By Schauder's Fixed Point Theorem there is a point z in D with $F(z) = z$ which gives $x_\alpha + \frac{r}{\|x_\alpha - Pz\|}[x_\alpha - Pz] = z$ and hence $x_\alpha = \frac{\|x_\alpha - Pz\|}{\|x_\alpha - Pz\| + r}z + \frac{r}{\|x_\alpha - Pz\| + r}Pz$. This implies that $Px_\alpha = Pz$, so $Pz = Px$. [We proved above that $Px_t = Px$ for some $t \geq 0$ implies that $Px_s = Px$ for $0 \leq s < t$.. This proof shows that $Px_\alpha = Pz$]. We now get $x_\alpha + \frac{r}{\|x_\alpha - Pz\|}[x_\alpha - Px] = z$ which can be written as $z = x_\lambda$ with $\lambda > \alpha$. This contradicts the definition of α. Remains to show that P is continuous. If P is not continuous we can find $\{x_n\}$ converging to x and $\epsilon > 0$ such that $\|Px_n - Px\| \geq \epsilon\ \forall n$. Now $d(x, C) \leq \|x - Px_n\| \leq \|x - x_n\| + \|x_n - Px_n\| = \|x - x_n\| + d(x_n, C) \to d(x, C)$. If y is a limit point of $\{Px_n\}$ then we get $d(x, C) \leq d(x, y) \leq d(x, C)$ which implies that $y = Px$, contradicting the fact that $\|Px_n - Px\| \geq \epsilon \ \forall n$.

Problem 135

In the literature there are two definitions of adjoint of an $n \times n$ matrix A. According to one definition the conjugate transpose of a matrix is called the adjoint. According to the other definition the (i,j) element of the adjoint matrix is the co-factor of $a_{i,j}$: the determinant of the matrix obtained by deleting the $i-th$ row and the $j-th$ column. Find all matrices for which the definitions lead to the same adjoint.

Let B be the adjoint according to the first definition and C according to the second. As is well known $CA = AC = (\det A)I$. If $B = C$ then we get $BA = AB = (\det A)I$. Since AB is non-negative definite we see that $\det A \geq 0$. Note that if $\det A = 0$ then $\|Ax\|^2 =< BAx, x >= 0$ for all x so A is the zero matrix. Otherwise, $\det A > 0$. Now, $\det(AB) = (\det A)^n \det(I)$ and this gives $|\det A|^2 = (\det A)^n$. If $n > 2$ this (along with $\det A > 0$) gives $\det A = 1$ and so $AB = BA = I$. Thus A is a unitary matrix with determinant 1. For $n > 2$ any unitary matrix with determinant 1 satisfies $B = C$ [Indeed $CA = AC = I$ and $BA = AB = I$ together imply $B = C$] and so does the zero matrix. It remains to consider the case $n = 2$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the condition is $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$. This says $c = -\bar{b}$ and $d = \bar{a}$. Thus $A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ with a and b arbitrary. Any such 2×2 matrix satisfies the condition $B = C$.

Problem 136

Let B be a bounded set in a Banach space X. Show that the following are equivalent:

a) B is an open ball

b) for any two points x, y in B there is an open ball V contained in B and containing x and y.

a) implies b): take $V = B$. Let b) hold. B is clearly open. Let $R = \sup\{r > 0$ such that there is an open ball of radius r contained in $B\}$. Let $r_n \uparrow R$ and $U_n = B(x_n, r_n) \subset B \ \forall n$. Let d_{nm} be the diameter of $U_n \cup U_m$. Claim: $d_{nm} \geq r_n + r_m + \|x_n - x_m\|$. To see this first assume $x_n \neq x_m$ and consider the points $x_n - \lambda(x_m - x_n)$ and $x_m + \mu(x_m - x_n)$ where $0 < \lambda < \frac{r_n}{\|x_n - x_m\|}$ and $0 < \mu < \frac{r_m}{\|x_n - x_m\|}$. These points belong to $U_n \cup U_m$ and the distance between them is $|1 + \lambda + \mu| \, \|x_n - x_m\|$. This number cannot exceed d_{nm}. Let $\lambda \to \frac{r_n}{\|x_n - x_m\|}$ and $\mu \to \frac{r_m}{\|x_n - x_m\|}$ to get $\left|1 + \frac{r_n}{\|x_n - x_m\|} + \frac{r_m}{\|x_n - x_m\|}\right| \|x_n - x_m\| \leq d_{nm}$. This proves the claim when $x_n \neq x_m$. The claim is trivial when $x_n = x_m$.

Let $\delta_n > 0$ and choose $u_n, v_n \in U_n \cup U_m$ with $\|u_n - v_n\| > r_n + r_m + \|x_n - x_m\| - \delta_n$. Let S_n be an open ball such that $u_n, v_n \in S_n$ and $S_n \subset B$. Then radius of S_n cannot exceed R. Hence $\|u_n - v_n\| \leq 2R$. Hence $r_n + r_m + \|x_n - x_m\| - \delta_n < 2R$. Since $r_n \uparrow R$ we see that $\{x_n\}$ is Cauchy. Let $x = \lim x_n$. We claim that $B = B(x, R)$. Since $\|y - x\| < R$ implies $\|y - x_n\| < r_n$ for n sufficiently large and since $B(x_n, r_n) \subset B$ we get $B(x, R) \subset B$. Suppose $\|y - x\| > R$. Then there is a point w in $B(x, R)[\subset B]$ such that $\|y - w\| > 2R$. [Take $w = x + t(x - y)$ where $\frac{2R}{\|y-x\|} - 1 < t < \frac{R}{\|y-x\|}$]. If $y \in B$ then there is an open ball inside B containing y and w. This ball has radius at most R. This contradicts the fact that $\|y - w\| > 2R$. Hence $y \notin B$. This proves that B is contained in the closure of $B(x, R)$. But B is open and hence $B \subset B(x, R)$.

Problem 137

Show that any (complex) polynomial p whose degree is $\leq n$ is the sum of three polynomials whose zeros are all on the real line.

Let $f(z) = \sum_{j=1}^{k} c_j z^j$ and $g(z) = \sum_{j=1}^{k} d_j z^j$ where $p(z) = \sum_{j=1}^{k} a_j z^j$ and $c_j = \operatorname{Re} a_j, d_j = \operatorname{Im} a_j$. Note that $k \leq n$. Let $C = \max\{\{\sup |f(x)| : -1 \leq x \leq n\}, \{\sup |g(x)| : -1 \leq x \leq n\}$. Let $p_0(z) = (1+i)Mz(z-1)(z-2)...(z-n+1)$ where $M > 4C$. Note that the zeros of p_0 are all real. Let $p_1(z) = f(z) - Mz(z-1)(z-2)...(z-n+1)$ and $p_2(z) = i[g(z) - Mz(z-1)(z-2)...(z-n+1)]$. Then $p_0 + p_1 + p_2 = f + ig = p$. Now $|Mz(z-1)(z-2)...(z-n+1)| > C$ at the points $z = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, ..., n - \frac{1}{2}$. The vaues of $Mz(z-1)(z-2)...(z-n+1)$ at these points alternate in sign. Also $|f(z)| \leq C$ and $|g(z)| \leq C$ at these points. It follows by intermediate value property that p_1 and p_2 both have one zeros between any two successive points in $\{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, ..., n - \frac{1}{2}\}$. These have to be only zeros because the degrees of these polynomials are n.

Problem 138

Show that any (complex) polynomial p whose degree is $\leq n$ is the sum of three polynomials whose zeros are all on the unit cricle $\{z : |z| = 1\}$.

Let $q(z) = (z - i)^n p(\frac{iz-1}{1+iz})$. Then q is a polynomial of degree at most n. By Problem 137 we can write q as $q_0 + q_1 + q_2$ where $q_j, j = 0, 1, 2$ are polynomials of degree at most n whose roots are all real and whose sum is q. Let $p_j(z) = i^{-n}(z-1)^n q_j(\frac{z+1}{i(1-z)}), j = 0, 1, 2$.

Remark: it can be shown that $z^2 + 4iz + 1$ *cannot* be written as the sum of two polynomials of degree at most 2 whose roots are all on the unit circle.

Problem 139

Let H be a Hilbert space and $\{x_1, x_2, ..., x_n\}$ a finite subset of H. Find explicitly the point x in H for which $\sum_{j=1}^{n} \|x - x_j\|^2$ is minimum.

The answer is $x = \frac{x_{1+}, x_{2+}, ... + x_n}{n}$. To prove this consider $H^n \equiv H \times H \times ... \times H$ (n factors) as a Hilbert space in a natural way: $< (x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) >= \sum_{j=1}^{n} < x_j y_j >$. Let C be the diagonal of H^n ($C = \{(x, x, ..., x) : x \in H\}$). There is a point $(x, x, ..., x)$ in C that is closest to $(x_1, x_2, ..., x_n)$. This point is characterized by the fact that $(x_1 - x, x_2 - x, ..., x_n - x)$ is orthogonal to D. Hence $\sum_{j=1}^{n} < x_j - x, y >= 0$ for all $y \in H$. This implies $< \frac{x_{1+}, x_{2+}, ... + x_n}{n} - x, y >= 0$ for all $y \in H$. Thus $x = \frac{x_{1+}, x_{2+}, ... + x_n}{n}$.

Problem 140

Let H and K be Hilbert spaces, $y_1, y_2, ..., y_n \in K$ and $A_1, A_2, ..., A_n$ be bounded operators from H to K. Show that $\sum_{j=1}^{n} \|A_j x - y_j\|^2$ is minimized at $x = x_0$ if and only if $\sum_{j=1}^{n} A_j^* A_j x_0 = \sum_{j=1}^{n} A_j^* y_j$. [$n = 1, A_1 = I$ reduces this to previous problem. If the positive operator $\sum_{j=1}^{n} A_j^* A_j$ is invertible then there is a unique x_0].

Define $T : H \to K^n$ be $T(x) = (A_1 x, A_2 x, ..., A_n x)$. The distance from $(y_1, y_2, ..., y_n)$ to $T(H)$ is minimized at $T(x_0)$ and hence $(y_1, y_2, ..., y_n) - T x_0$ is orthogonal to Tz for each z. This gives $\sum_{j=1}^{n} A_j^* A_j x_0 = \sum_{j=1}^{n} A_j^* y_j$. Converse also holds.

Problem 141

Consider the inequality $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \|x - y\|$ where $\|x\| \geq 1$ and $\|y\| \geq 1$. Is the inequality true in any inner product space space? Is it true in any normed linear space?

Yes. If C the closed unit ball then $\left\| \frac{x}{\|x\|} - x \right\| = \|x\| - 1 \leq \|x\| - \|u\| \leq \|x - u\|$ whenever $u \in C$, so $\frac{x}{\|x\|}$ is a point of best approximation (i.e. projection) of x onto C. In the case of an inner product space we have $\|x\| - \|u\| < \|x - u\|$

unless $x = \lambda(u-x)$ for some $\lambda \geq 0$ and this gives $u = \frac{x}{\|x\|}$. Now S3 in Problem 134 above completes the proof. Now consider the points $(\frac{1}{2}, \frac{3}{2})$ and $(1,1)$ in $\mathbb{R}^2$ under the norm $\|(x,y)\| = \max\{|x|, |y|\}$. We have $\left\| \frac{(\frac{1}{2},\frac{3}{2})}{\|(\frac{1}{2},\frac{3}{2})\|} - \frac{(1,1)}{\|(1,1)\|} \right\| = \left\|(\frac{1}{3}, 1) - (1,1)\right\| = \frac{2}{3}$ and $\left\|(\frac{1}{2}, \frac{3}{2})\right\| - \|(1,1)\| = \frac{1}{2}$. Thus the inequality does not hold in general.

[Remark: if the inequality holds with $y = -x$ then $\|x\| \geq 1$ and $\|y\| \geq 1$.

Also, a simple argument using triangle inequality show that if $\|x\| \geq 1$,$\|y\| \geq 1$ and either $\|x\| \geq 2$ or $\|y\| \geq 2$ then the inequality holds. In Studia Math. vol. 25, 1965, p.271-276 J J Schaffer has shown that the inequality holds only in Hilbert spaces and some two dimensional spaces. In particular the inequality characterizes Hilbert spaces in spaces of dimension greater than 2].

Problem 142

Let k be a positive integer and A, b be $n \times n$ matrices with $AB^k - B^k A = B$. Show that B is nilpotent.

We claim that $AB^{2^j k} - B^{2^j k} A = 2^j B^{(2^j - 1)k+1}$ for all non-negative integers j. For $j = 0$ this is the hypothesis. If it holds for j then we have $B^{2^j k} A B^{2^j k} - B^{2^{j+1} k} A = 2^j B^{(2^{j+1}-1)k+1}$ and $AB^{2^{j+1} k} - B^{2^j k} A B^{2^j k} = 2^j B^{(2^{j+1}-1)k+1}$. Adding these two equations we get $AB^{2^{j+1} k} - B^{2^{j+1} k} A = 2^{j+1} B^{(2^{j+1}-1)k+1}$. The induction argument is complete and the claim is proved. Now, $\left\|2^j B^{(2^j-1)k+1}\right\| \leq \left\|AB^{2^j k}\right\| + \left\|B^{2^j k} A\right\| \leq 2\|A\| \left\|B^{2^j k}\right\| \leq 2\|A\| \left\|B^{(2^j-1)k+1}\right\| \left\|B^{k-1}\right\|$. If $B^{(2^j-1)k+1}$ is not the zero matrix then we get $2^j \leq 2\|A\| \left\|B^{k-1}\right\|$ for all $j \geq 1$ which is a contradiction.

Problem 143

Let X be a n.l.s. and define S_x as $\{y \in X : \|x+y\|^2 = \|x\|^2 + \|y\|^2\}$. Show that the following are equivalent:

i) X is an inner product space

ii) for any $x \in X$, any $y \in S_x$ and any $a \in \mathbb{R}$ we also have $ay \in S_x$.

If i) holds then $S_x = \{y :< x, y >= 0\}$ so ii) is obvious. Let ii) hold. We claim that for any $x \in X$ and $y \in X$ there exists a real number a such that $x \in S_{ax+y}$. Note that we can take $a = 0$ if $x = 0$. For $x \neq 0$ fixed consider the continuous function $f(t) = \|(1+t)x+y\|^2 - \|tx+y\|^2 - \|x\|^2$. We can also write $f(t)$ as $(\|(1+t)x+y\| - \|tx+y\|)(\|(1+t)x+y\| + \|tx+y\|) - \|x\|^2$. If $t > 0$ then $\|(1+t)x+y\| - \|tx+y\| = \left\|(1+t)x + \frac{1}{1+t}y + \frac{t}{1+t}y\right\| - \|tx+y\| = \left\|x + \frac{1}{1+t}y\right\| + \left\|tx + \frac{t}{1+t}y\right\| - \|tx+y\|$ and $\left|\left\|tx + \frac{t}{1+t}y\right\| - \|tx+y\|\right| \leq \left\|\frac{1}{1+t}y\right\| \to 0$ as $t \to \infty$. Hence $\|(1+t)x+y\| - \|tx+y\| \to \|x\|$ as $t \to \infty$. Of course,

$\|(1+t)x+y\| + \|tx+y\| \to \infty$ as $t \to \infty$. It follows that $f(t) \to \infty$ as $t \to \infty$. For $t < -1$, $\|(1+t)x+y\| - \|tx+y\| = -\left\|x + \frac{1}{1+t}y\right\| + \left\|tx + \frac{t}{1+t}y\right\| - \|tx+y\|$. Since $\left|\left\|tx + \frac{t}{1+t}y\right\| - \|tx+y\|\right| \leq \left\|\frac{1}{1+t}y\right\|$ so we see that $f(t) \to -\infty$ as $t \to -\infty$. Hence there is a real number t such that $f(t) = 0$. This means $\|(1+t)x+y\|^2 = \|tx+y\|^2 + \|x\|^2$. We can take $a = t$ and so have proved our claim. It now follows from ii) that $-tx, (1-t)x$ and $-(1+t)x$ are all in S_{tx+y}. In other words, $\|y\|^2 = \|-tx\|^2 + \|tx+y\|^2$,

$\|(1-t)x+tx+y\|^2 = \|(1-t)x\|^2 + \|tx+y\|^2$ and $\|-(1+t)x+tx+y\|^2 = \|-(1+t)x\|^2 + \|tx+y\|^2$. Adding the last two equations we get $\|x+y\|^2 + \|x-y\|^2 = (1-t)^2\|x\|^2 + (1+t)^2\|x\|^2 + 2\|tx+y\|^2$. Since $\|tx+y\|^2 = \|y\|^2 - t^2\|x\|^2$ we get $\|x+y\|^2 + \|x-y\|^2 = 2(1+t^2)\|x\|^2 + 2\|y\|^2 - 2t^2\|x\|^2 = 2\|x\|^2 + 2\|y\|^2$. Thus the parallelogram law holds in X.

Problem 144

Let X, Y, Z be random variables on a probability space with $\alpha X + \beta Y + \gamma Z \overset{d}{=} U$ whenever the real numbers α, β, γ satisfy $\alpha^2 + \beta^2 + \gamma^2 = 1$, where U has uniform distribution on $(-1, 1)$. [$\overset{d}{=}$ stands for "equal in distribution"]. Show that $X^2 + Y^2 + Z^2 = 1$ almost surely.

It follows that the following random variables have uniform distribution on $(-1,1)$: $X, Y, Z, \frac{X+Y}{\sqrt{2}}, \frac{X-Y}{\sqrt{2}}$. Let $V = X^2 + Y^2 + Z^2$. We show that $EV = 1$ and $EV^2 = 1$. These two facts imply that V is almost surely constant and the constant must be 1, thus completing the proof. It is obvious that $EV = 1$ because $EU^2 = \frac{1}{3}$. The fact that $EV^2 = 1$ requires some computations: $EX^4 = EY^4 = \frac{1}{5}, 0 = \frac{E(X+Y)^4 - E(X-Y)^4}{4} = 2EX^3Y + 2EXY^3$,

$\frac{E(X+Y)^4}{4} = \frac{EX^4 + 4EX^3Y + 6EX^2Y^2 + 4EXY^3 + EY^4}{4} = \frac{EX^4 + 6EX^2Y^2 + EY^4}{4} = \frac{\frac{1}{5} + 6EX^2Y^2 + \frac{1}{5}}{4}$. Since $\frac{X+Y}{\sqrt{2}} \overset{d}{=} U$ we have $\frac{E(X+Y)^4}{4} = \frac{1}{5}$. Thus $\frac{1}{5} = \frac{\frac{1}{5} + 6EX^2Y^2 + \frac{1}{5}}{4}$ which implies $EX^2Y^2 = \frac{1}{15}$. Similarly $EZ^2Y^2 = \frac{1}{15}$ and $EX^2Z^2 = \frac{1}{15}$. Now $E(X^2 + Y^2 + Z^2 - 1)^2 = \frac{3}{5} + 1 - 2(\frac{1}{3}) - 2(\frac{1}{3}) - 2(\frac{1}{3}) + 2(\frac{1}{15}) + 2(\frac{1}{15}) + 2(\frac{1}{15}) = 0$.

Problem 145

Let $f : [0,1] \to [0, \infty)$ be a continuously differentiable function. Let L be the length of the graph of f and A the area under the graph. Show that $A + L > \pi/4$.

Let $g(x) = f(x) - f(1) + L$. Then $g(1) = L$ and the length of the graph of g (which is $\int_0^1 \sqrt{1 + (g'(x))^2}dx$) is also L. We claim that $\sqrt{1 + (g'(x))^2}dx \geq \sqrt{1 - x^2} + xg'(x)$ and that equality holds if and only if $(1 - x^2)(1 + (g'(x))^2) = 1$.

Once this is proved we get $L \geq \int_0^1 \sqrt{1-x^2}dx + \int_0^1 xg'(x)dx = \pi/4 + g(1) - \int_0^1 g(x)dx = \pi/4 + g(1) - \int_0^1 f(x)dx + f(1) - L = \pi/4 - A + f(1) \geq \pi/4$. If equality holds here then $f(1) = 0$ and $\sqrt{1+(g'(x))^2}dx \geq \sqrt{1-x^2} + xg'(x)$ for all x which implies $(1-x^2)(1+(g'(x))^2) = 1$ for all x. This gives $(g'(x))^2 = \frac{x^2}{1-x^2}$ $\forall x$. Equivalently$(f'(x))^2 = \frac{x^2}{1-x^2}$ $\forall x$. By continuity of the derivative we get $f'(x) = \frac{x}{\sqrt{1-x^2}}$ for all x or $f'(x) = -\frac{x}{\sqrt{1-x^2}}$ for all x. Thus $f(x) = c \pm \sqrt{1-x^2}$ where c is a constant (and $\pm$ sign is independent of x). Since $f(1) = 0$ we must have $c = 0$ and since f is given to be non-negative we have $f(x) = \sqrt{1-x^2}$ $\forall x$. However, for this function $A = \pi/4$ so $A + L > \pi/4$. It follows that $A+L > \pi/4$. To complete the proof we have to show that $\sqrt{1+t} \geq \sqrt{1-s}+\sqrt{ts}$ for $t \geq 0$ and $s \in [0,1]$ with equality if and only if $(1+t)(1-s) = 1$. Note that $(\sqrt{1+t}-\sqrt{1-s})^2 = (1-\sqrt{(1+t)(1-s)})^2 + st \geq st$ and equality holds if and only if $\sqrt{(1+t)(1-s)} = 1$ as required.

Problem 146

Show that a random variable X has a symmetric distribution if and only if $\int_0^\infty P\{|X-t| \leq a\}dt = a$ for all $a > 0$.

Note that if F is the distribution function of X then $\int_0^\infty [F(t+a)-F(t-a)]dt = \int_0^\infty P\{t-a < X \leq t+a\}dt = \int_0^\infty P\{|X-t| \leq a\}dt$ for all $a > 0$ since F has only countable number of discontinuities. Now

$$\int_0^T [F(t+a)-F(t-a)]dt = \int_a^{T+a} F(t)dt - \int_{-a}^{T-a} F(t)dt = \int_{T-a}^{T+a} F(t)dt - \int_{-a}^{a} F(t)dt \to$$

$2a - \int_{-a}^{a} F(t)dt$ as $T \to \infty$. It remains only to show that $\int_{-a}^{a} F(t)dt = a$ for all $a > 0$ if and only if X has a symmetric distribution. If the equation holds then differentiation w.r.t a yields $F(a) - F(-a) = 1$ at all but countable many points; in other words $P\{-X < -a\} = P\{X \leq -a\}$ at all but countable many points. Clearly this implies that $X \stackrel{d}{=} -X$. Conversely if $X \stackrel{d}{=} -X$ then $\int_{-a}^{a} F(t)dt = \int_{-a}^{0} F(t)dt + \int_0^a F(t)dt = \int_0^a F(-t)dt + \int_0^a F(t)dt = \int_0^a [F(t)+F(-t)]dt =$

$\int_0^a [1-P\{X=t\}]dt = a$ for all $a>0$. More generally X has a distribution that is symmetric about a real number b if and only if $\int_b^\infty P\{|X-t|\leq a\}dt = a$ for all $a>0$. [This follows by a simple change of variable].

Problem 147

Show that $\mathbb{R}^n$ cannot be written as the union of a family $\{D_i : i \in I\}$ of closed balls such that $D_i \cap D_j = \emptyset$ for $i \neq j$.

The interiors of the given disks contain points with rational coordinates. Since these interiors are disjoint it follows that the given collection is countable. Let this collection be re-written as $\{D_1, D_2, ...\}$. The intersection of any two of these disks has atmost one point. Points that belong to two of thse disks form a countable set, say $\{x_1, x_2, ...\}$. There is a line segment γ with one end in D_1 and the other end in D_2 containing none of the points $\{x_n\}$. [Fix y in D_1 and note that there are uncountable many line segments from this point to points in D_2 such that they have no common points except y]. Let $A = \gamma \backslash \bigcup_{n=1}^{\infty} D_n^0$. Since γ contains a point on ∂D_1 it follows that $A \neq \emptyset$. A is clearly closed. Since $\mathbb{R}^n = \bigcup_{n=1}^{\infty} D_n$ and γ intersects each D_n in at most two points we see that A is at most countable. If A has an isolated point a then there is an open segment J in γ containing no other point of A so $J \subset \bigcup_{n=1}^{\infty} D_n^0$. By connectedness $J \subset D_n^0$ for some n. But $a \in J$ and $a \in A \subset (\bigcup_{n=1}^{\infty} D_n^0)^c$. This contradiction shows that A is perfect, hence uncountable. Since $A \subset \mathbb{R}^n = \bigcup_{n=1}^{\infty} D_n$ and $A \subset (\bigcup_{n=1}^{\infty} D_n^0)^c$ we see that every point of A is in the boundary of some D_n and belongs to γ. The segment γ can have at most two points of the boundary of any D_n and this contradicts the fact that A is uncountable.

Problem 148*

Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) \equiv P(f(x), f(y))$ for some polynomial P in two variables with real coefficients.

Since $f(x) = P(f(x), f(0))$ it follows that the range of f is contained in the set of zeros of the polynomial $p(t) = P(t, f(0)) - t$. This forces f to be a constant unless $p(x) \equiv 0$. Also, $P(f(x), f(y)) \equiv P(f(y), f(x))$ for all $x.y$ which implies that $P(u,v) = P(v,u)$ for all $u, v \in f(\mathbb{R})$. Since $P(u,v) - P(v,u)$ is a polynomial in u vanishing throughout an open interval (if f is not a constant) for fixed $v \in f(\mathbb{R})$ it must be identically 0; repeating this argument with the second variable we see that $P(u,v) = P(v,u)$ for all $u, v \in \mathbb{R}$. Since $P(x, f(0)) \equiv x$ and $P(f(0), y) \equiv y$ this forces $P(u,v)$ to be $a(u+v) + buv + c$ for some constants a, b, c?????????. [This proof is copied from Amer. Math. Monthly. Why should P be of degree 1?]. Now $f(x+y) = a(f(x) + f(y)) + bf(x)f(y) + c$ for all x, y. Put $y = 0$ to get $f(x) = af(x) + bf(x)f(0) + c + af(0)$. If f is not a constant then $1 = a + bf(0)$ and $c + af(0) = 0$; we get $f(x+y) = (1 - bf(0))(f(x) + f(y)) + bf(x)f(y) - f(0) + bf^2(0)$ for all x, y. If $b = 0$ this gives $f(x+y) = f(x) + f(y) - f(0)$ which implies $f(x) - f(0) \equiv \alpha x$ for some $\alpha \in \mathbb{R}$. Thus, $f(x) \equiv \alpha x + f(0)$. If $b \neq 0$ then $f(0) = \frac{1-a}{b}$. Let $g(x) = bf(x) + a$. Then $g(x+y) \equiv g(x)g(y)$. Since g is not identically 0 it has no zeros. It follows that $g(x) \equiv e^{\beta x}$ for some real number β and we get $f(x) = \frac{e^{\beta x} - a}{b}$. Hence the only possibilities are $f(x) \equiv ax + b$ and $f(x) \equiv ae^{\beta x} + b$.

Problem 149

Chracterize all C^1 functions f from an open interval I in $\mathbb{R}$ into $\mathbb{R}$ such that f satisfies a differential equation of the type $f^{(n)} + g_{n-1}f^{(n-1)} + ... + g_1 f' + g_0 f = 0$ where the $g_i's$ are all continuous.

If f and its first $(n-1)$ derivates vanish at some point t then $f^{(n)}(t) = 0$ by the differential equation and this forces f to be identically 0 by a standard result in theory of ODE's. If this is not the case we can define $g_k(x) = -\frac{f^{(k)}(x) f^{(n)}(x)}{[\{f(x)\}^2 + \{f'(x)\}^2 + ... + \{f^{(n-1)}(x)\}^2]}$ so that the given equation holds.

Problem 150

Let $c_1, c_2, ..., c_N$ be distinct non-zero complex numbers. Show that $\sum\limits_{k=1}^{N} \frac{1}{c_k} \prod\limits_{j \neq k} \frac{1}{c_j - c_k} = \frac{(-1)^{N+1}}{c_1 c_2 ... c_N}$.

Let $f(z) = \prod\limits_{k=1}^{N} \frac{1}{z - c_k}$. We can write $f(z)$ as $\sum\limits_{k=1}^{N} \frac{a_k}{z - c_k}$ for some complex constants $a_1, a_2, ..., a_N$. [Proof: induction on N]. Now $a_k = \lim\limits_{z \to c_k} f(z)[z - c_k] = \prod\limits_{j \neq k} \frac{1}{c_j - c_k}$. Hence $\prod\limits_{k=1}^{N} \frac{1}{0 - c_k} = \sum\limits_{k=1}^{N} \frac{a_k}{0 - c_k}$.

Problem 151

Let d_1 and d_2 be two metrics on a set X such that any open ball w.r.t one contains an open ball w.r.t. the other. Does it follow that the metrics are equivalent (in the sense they have the same open sets)?

No! Let $X = \mathbb{R}, d_1(x,y) = |x-y|, d_2(x,y) = |x-y|$ if $x, y \in [0,\infty)$ or $x, y \in (-\infty, 0)$ and $d_2(x,y) = |x-y|+1$ if $x \in [0,\infty)$ and $y \in (-\infty,0)$ or $y \in [0,\infty)$ and $x \in (-\infty, 0)$. Writing B_1 and B_2 for open balls w.r.t d_1 and d_2 we have $B_2(x,r) \subset B_1(x,r)$ and, for $x \neq 0$, $B_1(x,\delta) \subset B_2(x,r)$ if we define δ as $\min\{r, |x|\}$. Let $\delta = r/2$ if $x = 0$. Then $B_1(\delta,\delta) \subset B_2(x,r)$. The interval $[0,1) = B_2(0,1)$ is open w.r.t. d_2 but not w.r.t. d_1.

Problem 152

Let $\sum\limits_{n=1}^{\infty} a_n$ be a convergent series of positive terms. If $b_n \uparrow 1$ in such a way that $[\log(n)][1-b_n]$ is bounded show that $\sum\limits_{n=1}^{\infty} a_n^{b_n} < \infty$.

$\sum\limits_{n=k}^{\infty} a_n^{b_n} \leq e^{2\Delta} \sum\limits_{n=k}^{\infty} (a_n)^{1-\Delta/\log(n)} (\frac{1}{n^2})^{\Delta/\log(n)}$ for k sufficiently large (because $b_n \geq 1-\Delta/\log(n)$ and $(\frac{1}{n^2})^{\Delta/\log(n)} \equiv e^{-2\Delta}$) where $\Delta = sup\{[\log(n)][1-b_n]\}$. Now $(\alpha)^{1-\Delta/\log(n)}(\beta)^{\Delta/\log(n)} \leq [1-\Delta/\log(n)]\alpha + [\Delta/\log(n)]\beta$ for any positive numbers α and β by convavity of logarithm. Hence $\sum\limits_{n=k}^{\infty} a_n^{b_n} \leq e^{2\Delta} \sum\limits_{n=k}^{\infty} \{[1 - \Delta/\log(n)]a_n + [\Delta/\log(n)]\frac{1}{n^2}\} < \infty$.

Problem 153

Let P and Q be orthogonal projections on finite dimensional complex Hilbert space H. Show that PQ is an orthogonal projection if and only if all eigen values of $P+Q$ belong to $\{0\} \cup [1,\infty)$.

PQ is a projection if and only if $PQ = QP$. Suppose this is the case. Then $P^2 - Q^2 = (P+Q)(P-Q) = (P-Q)(P+Q)$. Let $(P+Q)x = \lambda x, x \neq 0$. We have $(P-Q)x = (P^2-Q^2)x = (P-Q)(P+Q)x = \lambda(P-Q)x$. If $\lambda \neq 1$ this gives $(P-Q)x = 0$ or $Px = Qx$ so $Px = Qx = (\lambda/2)x$. But P and Q are idempotent so $\lambda = 0$ or $\lambda = 2$. Thus $\lambda \in \{0,1,2\} \subset \{0\} \cup [1,\infty)$.

Now suppose all eigen values of $(P+Q)$ belong to $\{0\} \cup [1,\infty)$. Let $(P+Q)x = \lambda x, x \neq 0$. Then $\lambda(Px - Qx) = (P-Q)(P+Q)x = (P-QP+PQ-Q)x$ and $(PQ-QP)x = (\lambda-1)(Px-Qx)$. Now $(P+Q)(P-Q)x = (P-PQ+QP-Q)x = (P-Q)x - (\lambda-1)(Px-Qx) = (2-\lambda)(Px-Qx)$. If $Px \neq Qx$ then x is an eigen vector of $(P+Q)$ with eigen value $2-\lambda$. In that case either $\lambda = 2$ or $2-\lambda \geq 1$, by hypothesis. Thus, $\lambda = 2$ or $\lambda \leq 1$. But if $\lambda \neq 0$ then we also have $\lambda \geq 1$ by hypothesis so $\lambda \in \{0,1,2\}$. In all these cases we claim that $PQx = QPx$. If

$\lambda = 1$ then $(PQ - QP)x = (\lambda - 1)(Px - Qx) = 0$. If $\lambda = 2$ then $Px + Qx = 2x$. Thus $2\|x\| \leq \|Px\| + \|Qx\| \leq \|x\| + \|x\| = 2\|x\|$ so equality holds throughout and $Px = cQx$ with $c \geq 0$. But then $2x = (1+c)Qx$ and $\frac{2}{1+c}$ must be 0 or 1. This forces c to be 1 and we get $Px = Qx = x$. So $PQx = QPx = x$. Let $\lambda = 0$. Then $Px = -Qx$. This implies $< Px, x >= - < Qx, x >$ which forces both sides to be 0. Thus $\|Px\|^2 = \|Qx\|^2 = 0$ and $Px = Qx = 0$ and $PQx = QPx = 0$. Of course, $Px = Qx$ then also $PQx = QPx$. Thus in all cases $PQx = QPx$ whenever x is an eigen vector of $(P + Q)$. But eigen vectors of $P + Q$ span the entire space H. [$P + Q$ is non-negative definite, hence diagonalizable]. Hence $PQ = QP$.

Problem 154

Let Ω be an open connected realtively compact subset of a metric space (X, d). Assume $\partial\Omega \neq \emptyset$. Let $f : \Omega \to \Omega$ be a continuous map such that its range $f(\Omega)$ is open. Show that $d(f(x_0), \partial\Omega) = d(x_0, \partial\Omega)$ for some $x_0 \in \Omega$.

Let $g(x) = d(f(x), \partial\Omega) - d(x, \partial\Omega)$. It suffices to show that g takes both positive and negative values on Ω. Let $v \in \bar{\Omega}$ with $d(v, \partial\Omega) = \max\{d(z, \partial\Omega) : z \in \Omega\}$. Clearly $v \in \Omega$ and $g(v) \leq 0$. Suppose $f(\Omega) = \Omega$. Let $f(w) = v$ with $w \in \Omega$. Then $g(w) = d(f(w), \partial\Omega) - d(w, \partial\Omega) = d(v, \partial\Omega) - d(w, \partial\Omega) \geq 0$. If $f(\Omega) \neq \Omega$ then there a point exists $y \in \Omega \cap \partial f(\Omega)$ and, clearly, $y \notin f(\Omega)$. [If $\Omega \cap \partial f(\Omega) = \emptyset$ then $f(\Omega)$ has no boundary points in Ω and so it is open and closed in Ω. But Ω is connected so $f(\Omega) = \Omega$ or $f(\Omega) = \emptyset$, a contradiction]. Since $y \in \partial f(\Omega)$ there exists a sequence $\{y_n\} \subset \Omega$ such that $f(y_n) \to y$. There is a subsequence y_{n_j} converging to some point u in $\bar{\Omega}$. If $u \in \Omega$ then $y = \lim f(y_{n_j}) = f(u) \in f(\Omega)$ contradicting the fact that $y \notin f(\Omega)$. Hence $u \in \partial\Omega$. Now $d(y, \Omega) > 0$ because Ω is open. Thus $d(f(y_{n_j}), \partial\Omega) \geq \frac{1}{2} d(y, \Omega)$ for j sufficiently large. Since $d(y_{n_j}, \partial\Omega) \to d(u, \partial\Omega) = 0$ it folows that $g(y_{n_j}) = d(f(y_{n_j}), \partial\Omega) - d(y_{n_j}, \partial\Omega) > 0$ for j sufficiently large.

Problem 155

Let P and Q be projections on $\mathbb{C}^n$. If $Tr(PQPQ) = Tr(PQ)$ show that PQ is a projection. [Tr stands for trace].

We prove that $Tr[\{PQ - (PQ)^*\}\{(PQ) - (PQ)^*\}^*] = 0$. This implies that $PQ = (PQ)^*$ so $PQ = Q^*P^* = QP$ and hence PQ is a projection. Now $Tr[\{PQ - (PQ)^*\}\{(PQ) - (PQ)^*\}^*] = Tr[PQQ^*P^* - PQPQ - Q^*P^*PQ + Q^*P^*PQ]$

$= Tr[PQQP - PQPQ - QPPQ + QPPQ] = Tr(PQ) - Tr(PQ) - Tr(PQ) + TR(PQ)) = 0$

Problem 156

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a differentiable map such that $\|F(x)\| \leq \|x\|$ for all $x \in \mathbb{C}^n$. Show that F is linear.

[Differentiability means existence of a linear map $L_x : \mathbb{C}^n \to \mathbb{C}^n$ (for any $x \in \mathbb{C}^n$) such that $\frac{\|F(x+h)-F(x)-L_x h\|}{\|h\|} \to 0$ as $\|h\| \to 0$].

Let $g(z) = F(a + zb), z \in \mathbb{C}$ with $a, b \in \mathbb{C}^n$ fixed. g is differentable. We claim that $g(z) = \lambda(a, b) + z\mu(a, b)$, for some $\lambda(a, b), \mu(a, b) \in \mathbb{C}$. Assuming this we get $F(zb) = \lambda(0, b) + z\mu(0, b)$ and we get $\lambda(0, b) = 0$ by putting $z = 0$. Thus $F(zb) = z\mu(0, b)$. In particular $F(b) = \mu(0, b)$ so $F(zb) = zF(b)$. Also $F(a) = g(0) = \lambda(0, b) + 0\mu(0, b)$ and hence $F(a + zb) = F(a) + z\mu(a, b)$. Multiplying by $\frac{1}{z}$ and letting $|z| \to \infty$ we get $F(b) = \lim_z F(\frac{a}{z} + b) = \mu(a, b)$. Thus $F(a + zb) = F(a)+zF(b)$ which completes the proof. To prove the claim let $h(z) = g(z)-g(0)$ and $\zeta = h(z + \xi w) - h(z) - \xi h(w)$ where $z, \xi, w \in \mathbb{C}$ are fixed. Consider the entire function $z \to< h(z), \zeta >$. It follows from hypothesis that $|< h(u), \zeta >| \leq \alpha + \beta |u|$ for all $u \in \mathbb{C}$. Hence $< h(u), \zeta >= cu + d$ for some constants c and d. Now $< \zeta, \zeta >=< h(z + \xi w) - h(z) - \xi h(w), \zeta >= c(z + \xi w) + d - (cz + d) - \xi(cw + d) = 0$. We have proved that $h(z+\xi w) - h(z) - \xi h(w) = 0$ for all choices of $z, \xi, w \in \mathbb{C}$ which proves that h is linear. Hence $g(z) = g(0) + h(z)$ is of the type $\lambda(a, b) + z\mu(a, b)$.

Problem 157

If T and S are commuting bounded operators on a complex Banach space X and $T \neq S$ show that $d(\sigma(T), \sigma(S)) \leq \|T - S\|$.

We prove the stronger result that for any $\lambda \in \sigma(T)$ there exists $\mu \in \sigma(S)$ such that $|\lambda - \mu| \leq \|T - S\|$. Let $V = I - (\lambda I - T)(\lambda I - S)^{-1}$. [If $\lambda \in \sigma(S)$ there is nothing to prove]. Then $V = (\lambda I-S)(\lambda I-S)^{-1}-(\lambda I-T)(\lambda I-S)^{-1} = (T-S)(\lambda I-S)^{-1}$. We claim that that if the result is false then the spectral radius of $(\lambda I - S)^{-1}$ is less than $\frac{1}{\|T-S\|}$. Once this is proved we can conclude (from the fact that $TS = ST$) that spectral radius of V does not exceed the product of the spectral radii of $(T - S)$ and $(\lambda I - S)^{-1}$ which is less than 1. This means $I - V$ is invertible and hence $(\lambda I - T)$ is invertible which is a contradiction. To prove the claim note that $|\lambda - \mu| > \|T - S\|$ for all $\mu \in \sigma(S)$ by assumption. If the claim is false then there exists τ such that $|\tau| \geq \frac{1}{\|T-S\|}$ and $\tau \in \sigma((\lambda I - S)^{-1})$. It follows that $I - \tau(\lambda I - S)$ is not invertible. Hence $\mu \equiv \lambda - \frac{1}{\tau} \in \sigma(S)$. This implies that $|\lambda - \mu| > \|T - S\|$ which means $\frac{1}{\tau} > \|T - S\|$, a contradiction.

Problem 158

If A and B are projections on $\mathbb{C}^n$ show that the following operators have the same range:

$AB - BA, ABA - BAB, (AB)^2 - (BA)^2$.

Let $C = A+B-I, D = A-B$. Then $DC = A^2+AB-A-BA-B^2+B = AB - BA$. Also $DC^2 = (A - B)(A + B + I - 2A - 2B + AB + BA) = (A -$

$B)(I-A-B+AB+BA)=ABA-BAB$ and $DC^3=(AB)^2-(BA)^2$. Thus, the ranges of the three operators $AB-BA, ABA-BAB, (AB)^2-(BA)^2$ are the images under D of the ranges of C, C^2 and C^3. However C is self-adjoint and hence diagonalizable. It follows from this that the ranges of C, C^2 and C^3 are all the same.

Problem 159

Prove or disprove the following:
if $f:\mathbb{R}\to\mathbb{R}$ is continuous then there exists $a\in\mathbb{R}$ such that $|f(a)|-|f(x)| < |a-x|$ for all $x\neq a$.

True! Since $|f(x)|+|x|/2\to\infty$ as $|x|\to\infty$ it attains its minimum value at some point a. For any $x\neq a$ we have $|f(x)|+|x|/2\geq|f(a)|+|a|/2$ and so $|f(a)|-|f(x)|\leq|x|/2-|a|/2\leq\frac{|a-x|}{2}<|a-x|$.

Problem 160

Let A and B be $n\times n$ matricies with real entries such that $A^2+B^2=AB-BA$. If $AB-BA$ is invertible show that 4 divides n.

We have $(A+iB)(A-iB)=A^2+B^2+iBA-iAB=(1-i)(AB-BA)$. Hence $(1-i)^n\det(AB-BA)\geq 0$. In particular $(1-i)^n$ is a real number. Thus $2^{n/2}e^{-in\pi/4}$ is real. Therefore $\sin(n\pi/4)=0$. QED

Problem 161

Let $f(x)=\log(1+x)$ for $x>0$. Let $f_1=f, f_{n+1}=f\circ f_n$ $(n\geq 1)$. Show that $f_n(x)\to\infty$ as $n\to\infty$ for each x and find the precise rate at which $f_n(x)\to\infty$.

We claim that $nf_n(x)\to\frac{2x}{x+2}$. We use the inequalities $\frac{2x}{x+2}\leq\log(1+x)\leq\frac{2x}{x+2-x^2}$ $\forall x\in(0,\infty)$. [For the right hand inequality consider the cases $x<1$ and $x\geq 1$ separately. For $x\geq 1$ we have $x\leq\frac{2x}{x+2-x^2}$]. Let $a_0=\frac{x}{n}, a_{n+1}=f(a_n)$ for $n\geq 0$. Then $\{a_n\}$ is decreasing sequence of positive numbers. Since $a_{n+1}\geq\frac{2a_n}{a_n+2}$ we get $\frac{1}{a_{n+1}}\leq\frac{1}{a_n}+\frac{1}{2}$. By iteration we get $\frac{1}{a_n}\leq\frac{n}{2}+\frac{1}{a_0}=\frac{n}{2}+\frac{n}{x}=n\frac{x+2}{2x}$. Hence $\liminf_n na_n\geq\frac{2x}{x+2}$. Now $a_{n+1}\leq\frac{2a_n}{a_n+2-a_n^2}$ and $\frac{1}{a_{n+1}}\geq\frac{a_n+2-a_n^2}{2a_n}=\frac{1}{2}+\frac{1}{a_n}-\frac{a_n}{2}\geq\frac{1}{2}+\frac{1}{a_n}-\frac{a_0}{2}$. By iteration $\frac{1}{a_n}\geq(n\frac{1-a_0}{2})+\frac{1}{a_0}=(n\frac{n-x}{2n})+\frac{n}{x}=\frac{nx-x^2+2n}{2x}$. Hence $na_n\leq\frac{2x}{x-x^2/n+2}$ and $\limsup_n na_n\leq\frac{2x}{x+2}$.

Problem 162

Let P and Q be projections onto closed subspaces M and N of a Hilbert space H. Find a necessary and sufficient condition on M and N for PQ to be a projection.

The condition is: $M\cap(M\cap N)^{\perp}$ and $N\cap(M\cap N)^{\perp}$ are orthogonal to each other. If PQ is a projection then $PQ=QP$. Let $x\in M\cap(M\cap N)^{\perp}$ and $y\in N\cap(M\cap N)^{\perp}$. Since PQ is the projection on $M\cap N$, we have $PQx=0$ and $QPx=0$. Thus, together with $PQ=QP$ gives $Qx=0$ and $Py=0$, i.e. $x\in N^{\perp}$ and $y\in M^{\perp}$. Since $x\in N^{\perp}$ and $y\in N$ we get $<x,y>=0$. This proves the necessity of the condition. Now suppose $M\cap(M\cap N)^{\perp}$ and $N\cap(M\cap N)^{\perp}$ are orthogonal to each other. Let $x\in H$. We can write x as $x_1+x_2+x_3+x_4$ where $x_1\in M\cap N, x_2\in M\cap N^{\perp}, x_3\in N\cap M^{\perp}$ and $x_4\in(M+N)^{\perp}$. This is because the spaces $M\cap N, M\cap(M\cap N)^{\perp}$ and $N\cap(M\cap N)^{\perp}$ are orthogonal to each other and their sum is $M+N$. We have $PQx_1=QPx_1=x_1$ and $PQx_4=QPx_4=0$. Also $PQx_2=QPx_2=PQx_3=QPx_3=0$. Thus $PQ=QP$ and hence PQ is a projection.

Problem 163

Let $f,g:[0,1]\to\mathbb{R}$ be continuous. If $\int\limits_0^1 fg=0$ show that $(\int\limits_0^1 g^2)(\int\limits_0^1 f^2)\geq 4[(\int\limits_0^1 f)(\int\limits_0^1 g)]^2$. Also show that $(\int\limits_0^1 g^2)(\int\limits_0^1 f)^2+(\int\limits_0^1 f^2)(\int\limits_0^1 g)^2\geq 4[(\int\limits_0^1 f)(\int\limits_0^1 g)]^2$.

Normalization reduces the proof of the first inequlaity to the case $\int\limits_0^1 f^2=\int\limits_0^1 g^2=1$. Since $\{f,g\}$ is orthonormal we can apply Bessel's inequlaity to the constant function 1 to get $(\int\limits_0^1 f)^2+(\int\limits_0^1 g)^2\leq 1$, This and the inequality $[(\int\limits_0^1 f)^2+(\int\limits_0^1 g)^2]^2\geq 4(\int\limits_0^1 f)^2(\int\limits_0^1 g)^2$ give $4(\int\limits_0^1 f)^2(\int\limits_0^1 g)^2\leq[(\int\limits_0^1 f)^2+(\int\limits_0^1 g)^2]^2\leq(\int\limits_0^1 f)^2+(\int\limits_0^1 g)^2\leq 1$ which gives the first inequality. for the second inequality we cannot assume that $\int\limits_0^1 f^2=\int\limits_0^1 g^2=1$. Let $a=\frac{(\int\limits_0^1 f)^2}{\int\limits_0^1 f^2}$ and $b=\frac{(\int\limits_0^1 g)^2}{\int\limits_0^1 g^2}$. We have to show that $\frac{1}{a}+\frac{1}{b}\geq 4$. If we show that $a+b\leq 1$ it would follow that $\frac{1}{a}+\frac{1}{b}\geq\frac{1}{a}+\frac{1}{1-a}\geq 4$ since $a(1-a)$ attains its maximum value on $[0,1]$ at the point $a=\frac{1}{2}$. For the inequality $a+b \leq 1$ note that $\frac{f}{(\int f^2)^{1/2}}$ and $\frac{g}{(\int g^2)^{1/2}}$ form an orthonormal set apply Bessel's inequlaity to the constant function 1.

Problem 164

Let P and Q be projections on $\mathbb{C}^n$. Show that any eigen value of $PQ+QP$ is $\geq -1/4$. Is $-1/4$ attained?

$0 \leq (P+Q-\frac{1}{2}I)^2 = P+Q+\frac{1}{4}I+PQ+QP-P-Q$ so $PQ+QP \geq -1/4I$. Hence any eigen value of $PQ+QP$ is $\geq -1/4$. Take $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}$ to show that $-1/4$ is attained. Indeed, $PQ+QP = \frac{1}{4}\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$ and eigen values of this matrix are $-1/4$ and $3/4$.

Problem 165

Let A be an $n \times n$ complex matrix such that $A^2 = 0$. Show that $R(A+A^*) = R(A)+R(A^*)$ where $R(.)$ is the range of $(.)$.

Let $N(.)$ be the kernel of $(.)$. By hypothesis, $R(A) \subset N(A)$. If $y \in R(A) +$

$R(A^*)$ then $y = Au + A^*v$ and we can decompose $u-v$ as x_1+x_2 where $x_1 \in R(A), x_2 \in N(A^*)$. (This is because $R(A) = (N(A^*))^{\perp}$). Note that $x_1 \in N(A)$ because $R(A) \subset N(A)$. Consider $w = u - x_1 = x_2 + v$. We have $(A+A^*)w = Aw + A^*w = A(u-x_1) + A^*(x_2+v) = Au - 0 + 0 + A^*v = y$. This completes the proof.

Problem 166

Let $f \in C[0,1]$ and $f(1) = 0$. Show that there exists $a \in (0,1]$ with $f(a) = \int_0^a f(x)dx$.

Let $g(x) = e^{-x}\int_0^x f(x)dx$. Then $g'(x) = e^{-x}f(x) - e^{-x}\int_0^x f(x)dx$. Suffices to show that there exists $a \in (0,1]$ with $g'(a) = 0$. If no such a exists then either g is decreasing on $(0,1)$ or increasing there. Let $h = g^2$. Then h is increasing (because $g > g(0) = 0$ if $g' > 0$ on $(0,1)$ and $g < g(0) = 0$ if $g' < 0$ on $(0,1)$ so that, in either case, $2gg' > 0$) but $h'(1) = 2g(1)g'(1) =$

$[2e^{-1}\int_0^1 f(x)dx][e^{-1}f(1)-e^{-1}\int_0^1 f(x)dx] = -2e^{-2}[\int_0^1 f(x)dx]^2 < 0$ where we used the fact that $f(1)=0$ and there would be nothing to prove if $\int_0^1 f(x)dx = 0$.

Problem 167

Prove that $\sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k} = \sum_{k\le n/2} \binom{n}{2k}\binom{2k}{k} 3^{n-2k}$ for any positive integer n.

Consider the coefficient of x^n in the polynomial $(1+3x+x^2)^n$. We have $(1+3x+x^2)^n = \sum_{k=0}^{n}\binom{n}{k}(1+x^2)^k(3x)^{n-k} = \sum_{k=0}^{n}\binom{n}{k}(3x)^{n-k}\sum_{j=0}^{k}\binom{k}{j}x^{2j}$. The coefficient of x^n in this is the right side of the identity we are required to prove. Now, $(1+3x+x^2)^n = [(1+x)^2+x]^n = \sum_{k=0}^{n}\binom{n}{k}(1+x)^{2k}x^{n-k}$

$= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}\sum_{j=0}^{2k}\binom{2k}{j}x^j$ and the coefficient of x^n is $\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}$, the left side of the identity.

Problem 168

Let $f,g:[0,1]\to\mathbb{R}$ be continuous. Prove that there exists $a\in(0,1)$ such that $\int_0^1 f(x)dx\int_0^a xg(x)dx = \int_0^1 g(x)dx\int_0^a xf(x)dx$.

Case 1: $\int_0^1 f(x)dx \neq 0 \neq \int_0^1 g(x)dx$. Let $h(x) = \frac{f(x)}{a} - \frac{g(x)}{b}$ where $a = \int_0^1 f(x)dx, b = \int_0^1 g(x)dx$. Let $H(x) = \int_0^x th(t)dt$. Claim: $H(1) + \int_0^1 \frac{H(x)}{x^2}dx = 0$. To see this note that $\int_0^1 \frac{H(x)}{x^2}dx = -\frac{1}{x}H(x)|_0^1 + \int_0^1 \frac{H'(x)}{x}dx = -H(1) + \int_0^1 h(x)dx = -H(1)$. This proves the claim and we conclude that H cannot be positive throughout $(0,1)$ or negative throughout $(0,1)$. Hence $H(a) = 0$ for some $a \in (0,1)$ which gives $\int\limits_0^1 f(x)dx \int\limits_0^a xg(x)dx = \int\limits_0^1 g(x)dx \int\limits_0^a xf(x)dx$.

Case 2: $\int\limits_0^1 f(x)dx = 0$. Let $H(x) = \int\limits_0^x tf(t)dt$. We have $\int\limits_0^1 \frac{H(x)}{x^2}dx = -\frac{1}{x}H(x)|_0^1 + \int\limits_0^1 \frac{H'(x)}{x}dx = -H(1) + \int\limits_0^1 f(x)dx = -H(1)$. As in the previuos case we conclude that $H(a) = 0$ for some $a \in (0,1)$ which gives $\int\limits_0^1 f(x)dx \int\limits_0^a xg(x)dx = \int\limits_0^1 g(x)dx \int\limits_0^a xf(x)dx = 0$. Similar argument works when $\int\limits_0^1 f(x)dx = 0$.

Problem 169

Prove or disprove that if A is set of (Lebesgue) measure 0 in $\mathbb{R}$ and $\epsilon > 0$ then there exist intervals $I_1, I_2, ...$ such that $A \subset \bigcup\limits_n I_n$ and the length of I_n does not exceed $\epsilon/2^n$ for any n.

False! If this is true then $\inf\{\sum\limits_{n=1}^{\infty}(diam(U_n)^p : U_n$ open and $A \subset \bigcup\limits_n I_n\} \leq \sum\limits_{n=1}^{\infty}(\epsilon/2^n)^p = \epsilon^p \sum\limits_{n=1}^{\infty}(1/2^n)^p \to 0$ as $\epsilon \to 0$ showing that the Hausdorff dimension of A is 0. The Cantor set is an example of a set of measure 0 whose Housdorff dimension $(\frac{\log 2}{\log 3})$ is positive.

Problem 170

Show that there is no continuous function $f : (0.\infty) \to \mathbb{R}$ such that $f(x) = 0 \Leftrightarrow f(2x) \neq 0$.

Let $A = \{x : f(x) = 0\}$. Then $A^c = 2A$. A is a closed subset of $(0.\infty)$ and hence $A^c = 2A$ is open. This implies that A is open and closed on the connected set $(0.\infty)$. So $A = \varnothing$ or $A = (0.\infty)$. But we cannot have $A^c = 2A$ in these cases.

Problem 171

Prove that $x \int\limits_x^{x+1} \sin(t^2)dt < 1$ for all $x > 1$.

Integrating by parts $\int_x^{x+1} \sin(t^2)dt = -\int_x^{x+1} \frac{1}{2t}[(-2t)\sin(t^2)]dt = -\frac{1}{2t}\cos(t^2)|_x^{x+1} - \int_x^{x+1} \frac{1}{2t^2}\cos(t^2)dt < \frac{1}{2(x+1)} + \frac{1}{2x} + \int_x^{x+1} \frac{1}{2t^2}dt = \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{2x} - \frac{1}{2(x+1)} = \frac{1}{x}$.

Problem 172

Let A be a compact subset of $\mathbb{R}$ and $P(A)$ the collection of all non-constant polynomials with real coefficients with leading coefficient 1. [Leading coefficient of $p(x)$ is the coefficient of the highest power x in p]. Let $\|p\|$ be $\sup\{|p(x)| : x \in A\}$.

a) Show that if there exists $p \in P(A)$ with $\|p\| < 2$ then there exists $p \in P(A)$ with $\|p\| < 1$

b) Show that if $A = [-2, 2]$ then there is no $p \in P(A)$ with $\|p\| < 2$.

a) Let $Sp(x) = p^2(x) - \frac{1}{2}\|p\|^2$. S maps $P(A)$ into itself and $\|Sp\| \leq \frac{1}{2}\|p\|^2$. Iteration gives $\|S^k p\| \leq \frac{1}{2^{k-1}}\|p\|^{2^k} \to 0$ as $k \to \infty$ if $\|p\| < 2$. Hence $\|S^k p\| < 1$ for k sufficiently large.

b) Suppose there exists $p_1 \in P(A)$ with $\|p_1\| < 2$. By a) there exists $p \in P(A)$ with $\|p\| < 1$. We claim the following:

i) there exists a map $T : P(A) \to P(A)$ such that $\|T\phi\| \leq \|\phi\|$ and if $\phi \in P(A)$ has degree $2k$ then $T\phi$ has degree k

ii) there exists $h \in P(A)$ such that $\deg(h) = 2^n$ for some n and $\|h\| < 2$. Note that $T^n h$ would then be an element of $P(A)$ with norm less than 2. This is a contradiction because $T^n h$ has degree 1. [$x + c$ has norm $2 + |c|$]. It reamins to construct T and h. Let $T\phi(x) = \phi_2(x+2)$ where $\phi_2(x^2) = \phi_1(x) = \frac{\phi(x)+\phi(-x)}{2}$. [ϕ_1 is a polynomial in x^2 and hence $\phi_2 \in P(A)$ exists]. This proves i). For ii) let $h(x) = x(g(x))^l$ where $g = T^m p$ and $\deg p = 2^m q, q$ odd and l is determined by the fact $q|(2^n - 1)$ so $l = \frac{2^n-1}{q}$ is an integer. Note that $\deg(h) = ql + 1 = 2^n$. Since $\|g\| \leq \|p\| < 1$ we have $\|h\| < 2$.

Problem 173

Let A be a discrete subset of $\mathbb{R}$. [i.e. $a \in A \Rightarrow \exists\ \delta > 0$ such that $A \cap (a - \delta, a + \delta) = \{a)$]. Can the closure of A be uncountable?

Yes! For each of the intervals removed in the construction of Cantor set pick a sequence increasing to the right end point and a sequence decresing to the left end point on that interval. Put all these sequences together to get a discrete set whose limit points include the end points of the intervals removed in the construction of Cantor set. The set of all these end ponts is the set of all finite sum $\sum_{n=1}^{N} \frac{a_i}{3^i}$ with $N \geq 1, a_i's \in \{0, 2\}$. It follows that every point of the Cantor set belongs to the closure of A.

Problem 174

Prove Banach's Theorem that any isometric map T from one normed linear X onto another normed linear Y with $T(0) = 0$ is linear.

Remark: the range of T must be a linear space for the proof to work.

We define the *centre* (x, y) of any two points x and y of X as follows: let $H_1 = \{z : d(z,x) = d(z,y) = \frac{1}{2}d(x,y)\}$ and, inductively, $H_n = \{z \in H_{n-1} : d(z,w) \leq \frac{1}{2}\delta(H_{n-1}) \ \forall w \in H_{n-1}\}$ where $\delta(A)$ stands for the diameter of the set A. We first prove that $\bigcap\limits_{n=1}^{\infty} H_n$ has at most one point. Note that $H_n \subset H_{n-1}$ and $\delta(H_n) \leq \frac{1}{2}\delta(H_{n-1})$: if $z_1, z_2 \in H_n$ then $z_2 \in H_{n-1}$ and hence $d(z_1,z_2) \leq \frac{1}{2}\delta(H_{n-1})$ by definition of H_n. It follows that $\bigcap\limits_{n=1}^{\infty} H_n$ has at most one point. Now, $v \equiv \frac{x+y}{2} \in H_1$. We claim that it belongs to each H_n. For this we verify the following:

$u \in H_n \Rightarrow x + y - u \in H_n$ $(n = 1, 2, ...)$ (1)

Indeed this result is trivial for $n = 1$ and we prove it by induction on n. Thus $u \in H_n \Rightarrow u \in H_{n-1} \Rightarrow x + y - u \in H_{n-1} \Rightarrow d(x + y - u, z) = d(x + y - z, u) \leq \frac{1}{2}\delta(H_{n-1})$ for all $z \in H_{n-1}$ because $u \in H_n$ and $x + y - z \in H_{n-1}$ by induction hypothesis. This proves (1). Now suppose $v \in H_{n-1}$. To show $v \in H_n$ we have to show $d(v,w) \leq \frac{1}{2}\delta(H_{n-1}) \ \forall w \in H_{n-1}$. But $d(v,w) = \left\|\frac{x+y}{2} - w\right\| = \frac{1}{2}\left\|x + y - 2w\right\| = \frac{1}{2}d(w, x + y - w) \leq \frac{1}{2}\delta(H_{n-1})$ since $x + y - w \in H_{n-1}$ by (1). We have proved that $v \in \bigcap\limits_{n=1}^{\infty} H_n$ and hence that $\bigcap\limits_{n=1}^{\infty} H_n = \{\frac{x+y}{2}\} \ \forall x, y \in X$. [We remark that this gives a definition of the centre or mid-point of x and y involving only the metric; algebraic operations are not involved!] It now follows easily that $T(\frac{x+y}{2}) = \frac{T(x)+T(y)}{2}$ for all $x, y \in X$. Since $T(0) = 0$ we get $T(\frac{x}{2}) = \frac{T(x)}{2}$ and hence $T(x + y) = T(x) + T(y)$. This and continuity of T yield linearity.

Problem 175

Let (X, d) be a metric space, $A \subset X$ and $f : X \to (0, \infty)$ a map such that $f(x)f(y) \leq d(x,y)$ whenever $x \in A$ and $y \in A^c$. Show that A and A^c are F_σ sets, i.e. they are countable unions of closed sets. Coversely, if A and A^c are F_σ sets show that such a function f exists.

[Remark: there is no function $f : \mathbb{R} \to (0, \infty)$ such that $f(x)f(y) \leq d(x,y)$ whenever $x \in Q$ and $y \in Q^c$ because Q^c is not an F_σ].

Let $A_n = \{a \in A : f(a) \geq \frac{1}{n}\}$. If $a \in A_n$ then $f(y) \leq nd(a,y)$ for $y \in A^c$. If $a \in \bar{A}_n \cap A^c$ then the same inequuality holds and when $y = a$ we get the contradiction that $f(a) = 0$. Thus $\bar{A}_n \subset A$. It follows that $A = \bigcup\limits_{n=1}^{\infty} \bar{A}_n$. Thus,

A is an F_σ. The hypothesis is symmetric in A and A^c, so A^c is also an F_σ. Now assume that A and A^c are F_σ sets. Let $\{C_n\}$ and $\{D_n\}$ be increasing sequences of closed sets with limits A and A^c respectively. For $x \in A$ let $N(x) = \min\{n \geq 1 : x \in C_n\}$. Let $f(x) = \min\{d(x, D_{N(x)}), 1\}$. Similarly for $y \in A^c$ let $M(y) = \min\{n \geq 1 : y \in D_n\}$ and let $f(y) = \min\{d(y, C_{M(y)}), 1\}$.. If $N(x) \geq M(y)$ then $y \in D_{M(y)} \subset D_{N(x)}$ so $f(x) \leq d(x, D_{N(x)}) \leq d(x,y)$ and $f(x)f(y) \leq f(x) \leq d(x,y)$. If $N(x) < M(y)$ then $f(x)f(y) \leq f(y) \leq d(y, C_{M(y)}) \leq d(x,y)$ because $x \in C_{N(x)} \subset C_{M(y)}$.

Problem 176

If $f : (0,\infty) \to \mathbb{R}$ is a measurable function such that $f(x+y)$ lies between $f(x)$ and $f(y)$ for all x and y show that f is a constant. Give an example of a non-constant (non-measurable) function with this property.

If f is not a constant then we can find x_1, x_2 such that $a_1 < a_2$ where $a_1 = f(x_1), a_2 = f(x_2)$. Let $S_1(y) = \{x \in (0,y) : f(x) \leq a_1\}$ and $S_2(y) = \{x \in (0,y) : f(x) \geq a_2\}$. If $x \in (0, \frac{1}{2}x_1)\backslash S_1(\frac{1}{2}x_1)$ then $f(x) > a_1$ and $f(x_1 - x) \leq a_1$; for, otherwise, $a_1 = f(x_1) = f(x + (x_1 - x))$ would be between $f(x)$ and $f(x_1 - x)$ and hence it would exceed a_1, a contradiction. Thus $x_1 - x \in S_1(x_1)$. This shows that $S_1(\frac{x_1}{2}) \cup [x_1 - \{(0, \frac{1}{2}x_1)\backslash S_1(\frac{1}{2}x_1)\}] \subset S_1(x_1)$. This implies that $m(S_1(x_1)) \geq \frac{x_1}{2}$ since $S_1(\frac{x_1}{2})$ and $x_1 - \{(0, \frac{1}{2}x_1)\backslash S_1(\frac{1}{2}x_1)\}$ are disjoint. [m is the Lebesgue measure]. Now note that $f(2x)$ lies between $f(x)$ and $f(x)$ so $f(2x) = f(x)$. By induction we get $f(nx) = f(x)$ for all $n \geq 1$ and for all $x \in (0,\infty)$. It follows that $f(qx) = f(x)$ for all rational $q > 0$ and for all $x \in (0,\infty)$. If $y > 0$ then we can find a rational number $q > 0$ such $qx_1 < y$. It follows that $qS_1(x_1) \subset S_1(y)$ and hence that $m(S_1(y)) \geq \frac{qx_1}{2}$. Letting $q \to \frac{y}{x_1}$ we get $m(S_1(y)) \geq \frac{y}{2}$. A similar argument shows that $m(S_2(y)) \geq \frac{y}{2}$. Since $S_1(y)$ and $S_2(y)$ are disjoint subsets of $(0,y)$ it follows that $m(S_1(y)) = \frac{y}{2}$ and $m(S_2(y)) = \frac{y}{2}$. This holds for each y and hence $m(S_1(y) \cap I) = \frac{m(I)}{2}$ for any interval $I \subset (0,1)$. It follows that the same holds for any measurable set $I \subset (0,1)$. In particular $m(S_1(1) \cap S_1(y)) = \frac{m(S_1(1))}{2}$ which means $m(S_1(1)) = 0$. Similarly $m(S_2(1)) = 0$. But $m(S_1(1)) = \frac{1}{2}$ which is a contradiction.

Let g be an additive non-measurable function: $\mathbb{R} \to \mathbb{R}$ and $f(x) = \frac{g(x)}{x}$. Then $f(x+y) = \frac{g(x)+g(y)}{x+y} = \frac{x}{x+y}f(x) + \frac{y}{x+y}f(y)$ which lies between $f(x)$ and $f(y)$ but f is not a constant.

Problem 177

Let C be a closed convex set in a normed linear space such that for some $\delta > 0$, $\|x\| \leq 1 + \delta$ implies $x = c + y$ with $c \in C$ and $\|y\| \leq 1$. Show that the interior of C is non-empty.

We claim that $\|x\| < \delta \Rightarrow x \in C$. If $a, b \in (0,\infty)$ we have $(a+b)C = aC + bC$. Writing B_r for $\{x : \|x\| \leq r\}$ we have $B_{1+\delta} \subset B_1 + C$. This gives $B_{(1+\delta)^2} =$

$(1+\delta)B_{1+\delta} \subset (1+\delta)B_1 + (1+\delta)C = B_{1+\delta} + (1+\delta)C \subset B_1 + C + (1+\delta)C = B_1 + (2+\delta)C$. By induction we get $B_{(1+\delta)^k} \subset B_1 + \sum\limits_{j=0}^{k-1}(1+\delta)^j C = B_1 + \frac{(1+\delta)^k-1}{\delta}C$. This gives $B_\delta \subset B_{\delta/t} + (1-\frac{1}{t})C$ where $t = (1+\delta)^k$. [t depends on k]. Now $\|x\| \leq \delta$ implies $x = x_k + y_k$ with $\|x_k\| \leq \delta/t$ and $y_k \in (1-\frac{1}{t})C$. Note that $x_k \to 0$ as $k \to \infty$. Also, $(1-\frac{1}{t}) \to 1$ so $\frac{1}{(1-\frac{1}{t})}y_k$ is a sequence in C converging to x. Since C is closed we see that $x \in C$.

Problem 178

Let $f : [0,\infty) \to \mathbb{R}$ be a C^∞ function such that $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in [0,\infty)$ and for all $n \geq 0$. Show that the function g defined by $g(x) = \frac{f(0)-f(x)}{x}(x > 0), g(0) = -f'(0)$ has the same property.

We have $xg^{(n)}(x) = -f^{(n)}(x) - ng^{(n-1)}(x)$ and $g^{(n)}(0) = -\frac{f^{(n+1)}(0)}{n+1}$. Hence $g^{(n-1)}(x) = -\frac{1}{x^n}\int\limits_0^x t^{n-1}f^{(n)}(t)dt$ and the result follows.

Problem 179

A Lemma in Rudin's real and Complex Analysis says that if $c_1, c_2, ..., c_N$ are complex numbers then we can find $S \subset \{1,2,...,N\}$ such that $\left|\sum\limits_{j\in S} c_j\right| \geq \frac{1}{\pi}\sum\limits_{j=1}^{N}|c_j|$. Prove that for any $\epsilon > 0$ we can find an example where $\left|\sum\limits_{j\in S} c_j\right| < (\frac{1}{\pi}+\epsilon)\sum\limits_{j=1}^{N}|c_j|$.

Let $c_j = e^{ij\pi/N}, 1 \leq j \leq N$. Then $\lim\limits_{N\to\infty} \frac{\left|\sum\limits_{j\in S(\theta)} c_j\right|}{\sum\limits_{j=1}^{N}|c_j|} = \lim\limits_{N\to\infty} \frac{\left|\sum\limits_{j\in S(\theta)} c_j\right|}{N} = \lim\limits_{N\to\infty} \frac{1}{2\pi} \frac{\left|\sum\limits_{j\in S(\theta)} c_j\right|}{N/2}\pi = \frac{1}{2\pi}\int\limits_{\theta-\pi/2}^{\theta+\pi/2} e^{it}dt = \frac{1}{\pi}$ where $S_\theta = \{j \in \{1,2,...,N\} : -\frac{\pi}{2} \leq \theta - \arg(c_j) \leq \frac{\pi}{2}\}$ (as in Rudin's book).

Problem 180

Let $\Im$ be the collection of all $N \times N$ matrices A such that $a_{ij} \geq 0, \sum_i a_{ij} = 1$ and $\sum_j a_{ij} = 1$ for all i, j. [Matrices of this type are called Doubly Stochastic]. Find all matrices in $\Im$ that commute with all other matrices in $\Im$.

Let A be such a matrix. Let π be a permutation of $\{1, 2, ..., N\}$. Let $P = ((p_{ij}))$ where $p_{ij} = 1$ if $j = \pi(i)$ and 0 otherwise. Then $P \in \Im$ and hence $AP = PA$. This gives $a_{i\pi^{-1}(j)} = a_{\pi(i)j}$ $\forall i, j$. In other words, $a_{ij} = a_{\pi(i)\pi(j)}$ $\forall i, j$, $\forall \pi$. This means that all the diagonal elements of A are the same and all the non-diagonal elements are the same. The converse is also true: if $a_{ii} = a$ and $a_{ij} = b$ for $i \neq j$ with $a + (N-1)b = 1$ [so that $A \in \Im$] then $AB = BA$ $\forall B \in \Im$.

Problem 181

Show that there is a continuous map $f : \mathbb{R} \to \mathbb{R}$ which is *not* 1-1 but 1-1 on Q. Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous map which is 1-1 on Q^c then it is 1-1 on $\mathbb{R}$.

First part is easy: $f(x) = (x - \sqrt{2})^2$ will do. Let (if possible) $f : \mathbb{R} \to \mathbb{R}$ be a continuous map which is 1-1 on Q^c but not 1-1 on $\mathbb{R}$. There exist real numbers a and b with $a < b$ and $f(a) = f(b)$. Let $A = \{x \in [a, b] : f(x) = f(a)\}$. If this set is dense in $[a, b]$ then f is a constant there which contradicts the hypothesis. Hence there is an interval $(\alpha, \beta) \subset [a, b]$ such that $f(x) \neq f(a)$ $\forall x \in (\alpha, \beta)$. We may suppose that (α, β) is the largest interval with this property so that either $f(\alpha) = f(a)$ and $f(\beta) = f(a)$. If f takes values greater than as well as less than $f(a)$ on (α, β) then it would take the value $f(a)$, a contradiction. Hence either $f((\alpha, \beta)) \subset (f(a), \infty)$ or $f((\alpha, \beta)) \subset (-\infty, f(a))$. These two cases are similar, so we assume that $f((\alpha, \beta)) \subset (f(a), \infty)$. Now $f([\alpha, \beta])$ is a compact interval containing $f(a)$ and contained in $[f(a), \infty)$ so it is of the type $[f(a), m]$ with $m > f(a)$. Also, $m = f(y)$ for some $y \in [\alpha, \beta]$. Now $f([\alpha, y]) = f([y, \beta]) = [f(a), m]$. [Indeed, $f([\alpha, y]) \subset f([\alpha, \beta]) \subset [f(a), \infty)$ and the supremum of $f([\alpha, \beta])$, namely m, belongs to $f([\alpha, y])$ so this interval has to be $[f(a), m]$. Similarly $f([y, \beta]) = [f(a), m]$]. If $x \in [\alpha, y]$ then $f(x) \in f([\alpha, y]) = f([y, \beta])$ so there exists $z \in [y, \beta]$ with $f(x) = f(z)$. Not both of x and z can be irrational (unless $x = y$) and we have a 1-1 map from the irrationals in $[\alpha, y]$ into the rationals in $[y, \beta]$ which is a contradiction.

Problem 182

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous. Prove that there is a non-empty proper closed C in $\mathbb{R}^2$ such that $f(C) \subset C$.

The proof below shows that $\mathbb{R}^2$ can be replaced by $\mathbb{R}^d$ for any $d \geq 1$. Let $C_x = \{x, f(x), f(f(x)), ...\}$. We prove that $\bar{C_x} \neq \mathbb{R}^2$ for some x. This would complete the proof since $f(C_x) \subset C_x$ and, by continuity, $f(\bar{C_x}) \subset \bar{C_x}$. Let $D = \{x : \|x\| \leq 1\}$ and $D_n = \{x \in D : \|f_j(x)\| \geq 1 \text{ for } 1 \leq j \leq n\}$ where $f_2 = f \circ f, f_3 = f \circ f \circ f$, etc. We consider two cases: $D_n = \emptyset$ for some n and $D_n \neq \emptyset$ for all n. In the first case take any $x \in D$ and note that $C_x \subset \bigcup_{j=1}^{n} f_j(D)$. [Let $x \in D$ and choose $1 \leq j_1 \leq n$ with $\|f_{j_1}(x)\| < 1$. Since $f_{j_1}(x) \in D$ there exists $1 \leq j_2 \leq n$ such that $\|f_{j_1+j_2}(x)\| < 1$. By induction we get $j_1, j_2, ... \in \{1, 2, ..., n\}$ such that $\|f_{j_1+j_2+...+j_k}(x)\| < 1$ for all k. Now any positive integer $m > n$ lies between $j_1 + j_2 + ... + j_k$ and $j_1 + j_2 + ... + j_{k+1}$ for some k and $f_m(x) = f_{m-(j_1+j_2+...+j_k)}(f_{j_1+j_2+...+j_k}(x)) \in \bigcup_{j=1}^{n} f_j(D)$ because $m - (j_1 + j_2 + ... + j_k) \leq j_{k+1} \leq n$]. Note that $\bar{C_x} \subset \bigcup_{j=1}^{n} f_j(D)$ because $f_j(D)$ is compact and $\bigcup_{j=1}^{n} f_j(D)$ is a proper subset of $\mathbb{R}^2$ because $\mathbb{R}^2$ is not compact. Now cosider the case when $D_n \neq \emptyset$ for all n. Each D_n is compact and hence $\bigcap_{n=1}^{\infty} D_n$ contains a point x_0. In this case $\bar{C_x}$ is disjoint from $\{x : \|x\| \leq \frac{\|x_0\|}{2}\}$ if $x_0 \neq 0$ and it is disjoint from $\{x : \frac{1}{3} \leq \|x\| \leq \frac{1}{2}\}$ if $x_0 = 0$. This proves that $\bar{C_x}$ is a proper subset of $\mathbb{R}^2$.

Problem 183

Let X and Y be random variables on a probability space such that $X, Y, X+Y$ and $X-Y$ all have the same distribution. Can we conclude that the common distribution is degenerate (at 0)? What if $EX^2 < \infty$? What if $E|X| < \infty$?

If $EX^2 < \infty$ then $E[(X+Y)^2+(X-Y)^2] = 2EX^2+2EY^2$ so $2EX^2 = 4EX^2$ so $X = 0$ a.s.. If $E|X| < \infty$ then $E|X| = E\left|\frac{X+Y}{2} + \frac{X-Y}{2}\right| \leq E\left|\frac{X+Y}{2}\right| + E\left|\frac{X-Y}{2}\right| = \frac{1}{2}E|X| + \frac{1}{2}E|X| = E|X|$ so $\frac{X+Y}{2} = Z\frac{X-Y}{2}$ for some non-negative r.v. Z. This gives $X = \frac{Z+1}{Z-1}Y$ and $|X| \geq |Y|$ (note that this last inequality holds even when $Z = 1$). But the hypothesis now implies that $|X| = |Y|$ a.s. which (in view of $X = \frac{Z+1}{Z-1}Y$ and $Z \geq 0$) implies $Z = 0$ and hence $\frac{X+Y}{2} = 0$ and so $\frac{X}{2} = 0$ a.s.! This finishes the case $E|X| < \infty$. In general, however, we cannot conlcude that the distribution is degenerate. Let U, V be i.i.d. with density $\frac{1}{\pi(1+x^2)}$. Let $X = \frac{U+V}{2}, Y = \frac{U-V}{2}$. Then $X, Y, X+Y, X-Y$ all have the charcteristic function $e^{-|t|}$.

Problem 184

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove or disprove that there exists a continuous strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ is differentiable on $\mathbb{R}$.

False. We prove that f is continuous and no-where differentiable thehn no such g exists. In fact $f \circ g$ cannot be differentiable on in any non-degenerate interval in $\mathbb{R}$. To see this assume thet $f \circ g$ is differentiable on (a, b) with $a < b$. Let $J = g(I)$ where $I = (a, b)$. Then $g^{-1} : J \to I$ is increasing, hence differentiable a.e.. Let y be a point in J at which g^{-1} is diffrentiable. Then $f = f \circ g \circ g^{-1}$ is differentiable at y because $f \circ g$ is differntiable at $g^{-1}(y)$. This is a contradiction.

Problem 185

Does there exist a dense set E in $\mathbb{R}^2$ such that every point in E has both coordinates rational but the distance between any two points of E is irrational?

Yes! Let $\{(x_n, y_n) : n = 1, 2, ...\}$ be dense in $\mathbb{R}^2$. Choose odd positive integers k_n, m_n such that $\left|x_n - \frac{k_n}{2^n}\right| \leq \frac{1}{2^n}$ and $\left|y_n - \frac{m_n}{2^n}\right| \leq \frac{1}{2^n}$. [The interval $[2^n x_n - 1, 2^n x_n + 1]$ contains an odd integer k_n]. The set $\{(\frac{k_n}{2^n}, \frac{m_n}{2^n}) : n = 1, 2, ...\}$ is dense in $\mathbb{R}^2$. [Let $(x, y) \in \mathbb{R}^2$ and $\delta > 0$. The open ball with center (x, y) and radius $\delta/2$ contains infinitely many of the points (x_n, y_n) and hence we can choose a point (x_n, y_n) in it with $\frac{1}{2^n} < \delta/2\sqrt{2}$. In this case the distance from (x, y) to $(\frac{k_n}{2^n}, \frac{m_n}{2^n})$ does not exceed $\delta/2 + \sqrt{\frac{1}{4^n} + \frac{1}{4^n}} = \delta/2 + \frac{\sqrt{2}}{2^n} < \delta$]. We complete the proof by showing that the distance between any two of the points $(\frac{k_n}{2^n}, \frac{m_n}{2^n})$ is irrational. Let $1 \leq j < n$. Then $4^n d^2((\frac{k_j}{2^j}, \frac{m_j}{2^j}), (\frac{k_n}{2^n}, \frac{m_n}{2^n})) = 4^n(\frac{k_j}{2^j} - \frac{k_n}{2^n})^2 + 4^n(\frac{m_j}{2^j} - \frac{m_n}{2^n})^2 = 4^{n-j}(k_j^2 + m_j^2) + k_n^2 + m_n^2 - 2^{n-j+1}k_j k_n - 2^{n-j+1}m_j m_n \equiv 2 \bmod(4)$ and hence $d((\frac{k_j}{2^j}, \frac{m_j}{2^j}), (\frac{k_n}{2^n}, \frac{m_n}{2^n}))$ is irrational since $\sqrt{\frac{2+4m}{4^n}} = (\sqrt{2+4m})/2^n$ is irrational for any two positive integers n and m. [If $\sqrt{2+4m} = \frac{p}{q}$ with $(p, q) = 1$ then p is even and q is odd. If $p = 2p_1$ we get $(2+4m)q^2 = 4p_1^2$ so $2|(1+2m)q^2$ which is absurd].

Problem 186

Does there exist a dense subset S of the unit circle S^1 such that all points in S have rational coordinates and the distance between any two points of S is rational?

Yes. One such set is $\{(\frac{4t(t^2-1)}{(t^2+1)^2}, \frac{4t^2-(t^2-1)^2}{(t^2+1)^2}) : t \in Q\}$.

Problem 187

There is no function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 2 \ \forall x \in \mathbb{R}$.

Write f_n for the $n-th$ iterate of f. Let A and B be the fixed points of f_2 and f_4 respectively. Then $A = \{x : x^2 - 2 = x\} = \{-1, 2\}$ and $B = \{x : x^4 - 4x^2 + 2 = x\} = \{-1, 2, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\}$. [To solve the fourth degree equation use the fact that two of th roots are -1 and 2]. Claim: f is a bijection of B.

For this first note that $b \in B \Rightarrow f(b)$ is also a fixed point of f_4 so $f(b) \in B$. If $b_1, b_2 \in B$ and $f(b_1) = f(b_2)$ then $f_4(b_1) = f_4(b_2)$ which yields $b_1 = b_2$. This proves the claim. Next we observe that f maps A into A into itself. Indeed $x \in A \Rightarrow f_2(x) = x$

$\Rightarrow f_2(f(x)) = f(f_2(x)) = f(x)$ so $f(x)$ is also a fixed point of f_2. This shows that f maps A into itself. [It is also clear that $f(-1) \neq f(2)$: otherwise $f_2(-1) \neq f_2(2)$ which says $-1 = 2$ a contradiction. Thus f is a bijection of A]. It now follows that $f(\frac{-1+\sqrt{5}}{2}) = \frac{-1\pm\sqrt{5}}{2}$. We cannot have $f(\frac{-1+\sqrt{5}}{2}) = \frac{-1+\sqrt{5}}{2}$ because $\frac{-1+\sqrt{5}}{2}$ would then be a fixed point of f_2. We have proved that $f(\frac{-1+\sqrt{5}}{2}) = \frac{-1-\sqrt{5}}{2}$ and $f(\frac{-1-\sqrt{5}}{2}) = \frac{-1+\sqrt{5}}{2}$. Thus $f_2(\frac{-1+\sqrt{5}}{2}) = \frac{-1+\sqrt{5}}{2}$ and $\frac{-1+\sqrt{5}}{2} \in A$ a contradiction.

Problem 188

Let $\mu_1, \mu_2, ..., \mu_n$ be non-atomic probability measures on $(\Omega, \mathcal{F})$. Show that there exist disjoint sets $A_1, A_2, ..., A_n$ in $\mathcal{F}$ such that $\mu_i(A_i) = \frac{1}{n}$ $(1 \leq i \leq n)$.

[See also Problem 295]

Suppose this holds for n non-atomic p.m.'s. Consider $(n+1)$ non-atomic p.m.'s $\mu_1, \mu_2, ..., \mu_{n+1}$. There exist disjoint sets $B_1, B_2, ..., B_n$ such that $\mu_i(B_i) \geq \frac{1}{n}$ $(1 \leq i \leq n)$ and $\bigcup\limits_{i=1}^{n} B_i = \Omega$. For each $i \leq n$ we can write B_i as a disjoint union of sets $B_{i,1}, ..., B_{i,n+1}$ with $\mu_i(B_{i,j}) = \frac{1}{n(n+1)}, 1 \leq j \leq n+1$. Arrange the sets$B_{i,1}, ..., B_{i,n+1}$ in such a way that $\mu_{n+1}(B_{i,1}) = \max\{\mu_{n+1}(B_{i,j}) : 2 \leq j \leq n+1\}$. Let $A_i = \bigcup\limits_{j=2}^{n+1} B_{i,j}$ $(1 \leq i \leq n)$ and $A_{n+1} = \bigcup\limits_{j=1}^{n} B_{i,1}$. Then $\mu_i(A_i) = \sum\limits_{j=2}^{n+1} \mu_i(B_{i,j}) = \sum\limits_{j=2}^{n+1} \frac{1}{n(n+1)} = \frac{1}{n+1}$ for $1 \leq i \leq n$. Also, $\mu_{n+1}(A_{n+1}) = \sum\limits_{i=1}^{n} \mu_{n+1}(B_{i,1}) \geq \sum\limits_{i=1}^{n} \frac{1}{n+1}\mu_{n+1}(B_i) = \frac{1}{n+1}\mu_{n+1}(\bigcup\limits_{i=1}^{n} B_i) = \frac{1}{n+1}$. This proves (by induction) that for each n there exist disjoint sets $A_1, A_2, ..., A_n$ in $\mathcal{F}$ such that $\mu_i(A_i) \geq \frac{1}{n}$ $(1 \leq i \leq n)$. Of course we can replace $A_i's$ by subsets so that $\mu_i(A_i) = \frac{1}{n}$ $(1 \leq i \leq n)$.

Problem 189

Let $\{X_i : i \in I\}$ be a family of random variables with finite mean. Which of the following condition imply which others?

a) $\{X_i : i \in I\}$ is uniformly integrable

b) There is an integrable random variable Y such that $|X_i| \leq Y$ a.s. for all $i \in I$

c) There is an integrable random variable Y such that $P\{|X_i| \geq a\} \leq P\{Y \geq a\}$ for all $i \in I$, for all $a \in [0, \infty)$.

b) implies c) .Also c) implies a). [In fact a simple Fubini argument shows $\int\limits_{\{|X_i|>\Delta\}} |X_i|\, dP \leq \int\limits_{\{Y>\Delta\}} Y dP$]. We give an example to show that a) does not imply c). This also implies that a) does not imply b). Let $I = (e, \infty)$ and X_i take values i and 0 with probabilities $\frac{1}{i \log i}, 1 - \frac{1}{i \log i}$. Then $\int\limits_{\{|X_i|>\Delta\}} |X_i|\, dP = 0$ or $\frac{1}{\log i} < \frac{1}{\log \Delta}$ according as $i \leq \Delta$ or $i > \Delta$. Thus a) holds. Suppose there is an integrable random variable Y such that $P\{|X_i| \geq a\} \leq P\{Y \geq a\}$ for all $i \in I$, for all $a \in [0, \infty)$. Then $P\{Y \geq a\} \geq P\{|X_i| \geq a\} = \frac{1}{i \log i}$ if $0 < a \leq i$. In particular $P\{Y \geq i\} \geq \frac{1}{i \log i}$ $\forall i \in (e, \infty)$. Hence $\int Y dP = \sum\limits_{k=0}^{\infty} \int\limits_{\{k \leq Y < k+1\}} Y dP \geq$ $\sum\limits_{k=0}^{\infty} kP\{k \leq Y < k+1\} = \sum\limits_{k=1}^{\infty} P\{Y \geq k\} \geq \sum\limits_{k=3}^{\infty} \frac{1}{i \log i} = \infty$. It remains to see if c) implies b). Let $\{\xi_n\}$ be i.i.d. $N(0, 1)$ and note that c) holds with $Y = \xi_1$. There is no random variable Z such that $|\xi_i| \leq Z$ a.s. for all i. This is clear from the fact that $P\{\sup\limits_i |\xi_i| \leq n\} = 0$ for each n so $Z > n$ a.s. for each n!

Problem 190

Does there exist a compact set K in a normed linear space X such that every point in $X \backslash K$ has exactly two points in K closest to it?

N0! We prove that if there is a point y such that y has exactly two points in K closest to it then there is a point that has a unique element in K that is closes to it. Let $y \in X \backslash K$ and choose $x \in K$ such that $d(y, K) = \|x - y\|$. Let $u = \frac{x+y}{2}$. Note that $B(y, \|x - y\|) \cap K = \emptyset$. Now the ball $B(u, \frac{\|x-y\|}{2}) \equiv B(u, \|x - u\|)$ is contained in $B(y, \|x - y\|)$ and hence it does not intersect K. It follows that $\|x - u\| = d(u, K)$. Now suppose there is another element $v \in K$ with $\|v - u\| = d(u, K)$. Then $\|v - y\| \leq \|v - u\| + \|y - u\| = d(u, K) + \|y - u\| \leq \|u - x\| + \|y - u\| = \frac{\|x-y\|}{2} + \frac{\|x-y\|}{2} = \|x - y\|$. It follows that v is the other point closest to y. Now $\|u - v\| = d(u, K) \leq \|u - x\| = \frac{\|x-y\|}{2}$. But then $\|y - v\| \leq \|u - y\| + \|u - v\| \leq \frac{\|x-y\|}{2} + \frac{\|x-y\|}{2} = \|x - y\|$. The fact that x and v are both at the same distance (viz. $d(y, K)$) from y shows that equality holds throughout and hence that $(u - y) = \alpha(u - v)$ with $\alpha \geq 0$. This gives $\alpha > 0$ and $v = \frac{\alpha-1}{2\alpha}x + \frac{\alpha+1}{2\alpha}y$. This and the fact that $\|y - v\| = \|x - y\|$ show that $\alpha = \frac{1}{3}$ and $v = 2y - x$. This contradicts $\|v - u\| = d(u, K)$ since the right side does not exceed $\|u - x\| = \frac{\|x-y\|}{2}$ where as the left side is $\|2y - x - u\| = \frac{3\|x-y\|}{2}$.

Problem 191

If $f : (0,1)\times(0,1) \to \mathbb{R}$ is separately continuous and if f vanishes on a dense subset then it is identically 0.

Suppose $f(a,b) > 0$ for some a and $b \in (0,1)$. There exists $\delta > 0$ such that $f(x,b) \geq \frac{1}{4}f(a,b)$ for $|x-a| \leq \delta$. Let $J = [a-\delta, a+\delta] \cap (0,1)$. Let $T_n = \{x \in J : f(x,y) \geq \frac{1}{4}f(a,b)$ whenever $|y-b| < \frac{1}{n}\}$. Then $J = \bigcup\limits_{n=1}^{\infty} T_n$ and each T_n is closed. By Baire Category Theorem there is an integer m such that T_m has non-empty interior. This implies that $T_m \times (b-\frac{1}{m}, b+\frac{1}{m})$ has non-empty interior in $(0,1)\times(0,1)$. By hypotheis there is a point (s,t) in $T_m \times (b-\frac{1}{m}, b+\frac{1}{m})$ such that $f(s,t) = 0$. By definition of T_m we get $f(s,t) \geq \frac{1}{4}f(a,b) > 0$ a contradiction. Similarly, $f(a,b) < 0$ for some a and $b \in (0,1)$ leads to a contradiction.

Problem 192

Compute $\sup\{\inf\{\frac{f(x)}{x}\int\limits_0^x \{1-f(t)\}dt : x > 0\} : f : [0,\infty) \to \mathbb{R}$ is continuous$\}$. Find all continuous functions f such that the supremum is attained at f.

The supremum is $\frac{1}{4}$ and it is attained only at the function $f(x) = \frac{1}{2}$ $\forall x \in [0,\infty)$. Proof: write $\sigma(f)$ for $\inf\{\frac{f(x)}{x}\int\limits_0^x \{1-f(t)\}dt : x > 0\}$. By MVT applied to $\int\limits_0^x \{1-f(t)\}dt$ we have $\frac{1}{x}\int\limits_0^x \{1-f(t)\}dt = 1 - f(g(x))$ with $0 < g(x) < x$. Since $\frac{f(x)}{x}\int\limits_0^x \{1-f(t)\}dt = f(x)\{1-f(g(x))\} \to f(0)\{1-f(0)\} \leq \frac{1}{4}$ as $x \to 0$ we get $\sigma(f) \leq \frac{1}{4}$. The desired supremum is therefore $\leq \frac{1}{4}$ and $\sigma(\frac{1}{2}) = \frac{1}{4}$, so the exact value of the supremum is $\frac{1}{4}$. Suppose f is a continuous function on $[0,\infty)$ with $\sigma(f) = \frac{1}{4}$. We will show that $f(x) = \frac{1}{2}$ $\forall x \in [0,\infty)$. Let $F(x) = \begin{cases} \frac{1}{x}\int\limits_0^x f(t)dt & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$. Note that $\sigma(f) = \frac{1}{4}$ implies $f(0)\{1-f(0)\} = \frac{1}{4}$ which implies $f(0) = \frac{1}{2}$. Thus F is continuous on $[0,\infty)$. Claim: $f(x) \geq F(x) \geq 0$ $\forall x$ and F is non-decreasing on $[0,\infty)$. Once the claim is proved we can complete the proof as follows: $f(x)[1-F(x)] = f(x)\frac{1}{x}\int\limits_0^x [1-f(t)]dt \geq \frac{1}{4}$ (because $\sigma(f) = \frac{1}{4}$). So $f(x) > 0$ and $F(x) < 1$. Let $l = \lim\limits_{x\to\infty} F(x) (\in [0,1])$. Note that if $\liminf\limits_{x\to\infty} f(x) >$

l then $F(x) = \frac{1}{x}\int\limits_0^x f(t)dt \geq \frac{1}{x}\int\limits_0^\Delta f(t)dt + \frac{1}{x}\int\limits_\Delta^x f(t)dt > \frac{1}{x}\int\limits_0^\Delta f(t)dt + \frac{1}{x}\int\limits_\Delta^x (l+\delta)dt$

$= \frac{1}{x}\int\limits_0^\Delta f(t)dt + \frac{1}{x}(l+\delta)(x-\Delta) \geq \frac{1}{x}(l+\delta)(x-\Delta) > l+\delta/2$ with sufficiently large Δ and x (and some $\delta > 0$) which is a contradiction. This proves that $\liminf\limits_{x\to\infty} f(x) \leq l$. However $f(x) \geq F(x) \to l$ so $\liminf\limits_{x\to\infty} f(x) = l$. Hence there is a sequence $x_n \to \infty$ such that $f(x_n) \to l$. But then $\frac{1}{4} = \sigma(f) \leq \liminf \frac{f(x_n)}{x_n}\int\limits_0^{x_n}\{1-f(t)\}dt = l(1-l) \leq \frac{1}{4}$ and hence $l = \frac{1}{2}$. But $F(0) = \frac{1}{2}, F \uparrow$ and $F(\infty) = \frac{1}{2}$ so $F(x) = \frac{1}{2}$ for all x. So $\int\limits_0^x f(t)dt = \frac{x}{2}$ for all $x > 0$ which gives $f(x) \equiv \frac{1}{2}$. It remains to prove the claim. Let $f \geq 0$ on $[0,c]$. (Such a $c > 0$ exists because $f(0) = \frac{1}{2}$). Since $f(x)[1-F(x)] \geq \sigma(f) = \frac{1}{4}$ and $f(x)[1-f(x)] \leq \frac{1}{4}$ we get $f(x)[f(x) - F(x)] \geq 0$ so $f(x) \geq F(x)$. [If $f(x) = 0$ then $\sigma(f) \leq 0$, a contradiction). Now $F'(x) = \frac{f(x)}{x} - \frac{1}{x^2}\int\limits_0^x f(t)dt = \frac{f(x)}{x} - \frac{1}{x^2}[xF(x)] = \frac{f(x)-F(x)}{x} \geq 0$ so F is non-decreasing on $[0,\infty)$. Since $F(0) = \frac{1}{2}$ it follows that $F(x) \geq 0$ for all x. The claim is now proved.

Problem 193

Let P and Q be projections on a Hilbert space H. It is well known that $P \leq Q$ in the sense $< Px, x > \leq < Qx, x >$ for all $x \in H$ if and only if $P = PQ = QP$. Let $P\blacktriangledown Q$ be the glb of P and Q, i.e. the largest projection R which is $\leq P$ and $\leq Q$. If $P+Q$ is invertible show that $P\blacktriangledown Q = 2P(P+Q)^{-1}P$.

Let $R = 2P(P+Q)^{-1}Q$. Then $R = 2P(P+Q)^{-1}(P+Q-P) = 2P - 2P(P+Q)^{-1}P = 2P[I-(P+Q)^{-1}]P$. This gives $RP = PR = R$. We also have $R = 2(P+Q-Q)(P+Q)^{-1}Q = 2Q - 2Q(P+Q)^{-1}Q = 2Q[[I-(P+Q)^{-1}]Q$ and so $RQ = QR = R$. Thus $2R = (P+Q)R = R(P+Q)$. This gives $R^2 = 2RP(P+Q)^{-1}Q = PR(P+Q)(P+Q)^{-1}Q = PRQ = R$. The formula $R = 2P[I-(P+Q)^{-1}]P$ shows that R is self adjoint. Hence R is a projection. Since $RP = PR = R$ and $RQ = QR = R$ we see that $R \leq P$ and $R \leq Q$. Suppose S is a projection such that $S \leq P$ and $S \leq Q$. We have to show that $S \leq R$. We have $2S = (P+Q)S = S(P+Q)$ and $RS = 2P(P+Q)^{-1}QS = 2P(P+Q)^{-1}S = P(P+Q)^{-1}[(P+Q)S] = PS = S$. Similarly $SR = 2SP(P+Q)^{-1}Q = 2S(P+Q)^{-1}Q = S(P+Q)(P+Q)^{-1}Q = SQ = S$. This completes the proof.

Problem 194

Let V and W be vector spaces and $T, S : V \to W$ be linear. Suppose for each $x \in V$ there is a scalar c_x such that $Tx = c_x Sx$ $(x \in V)$. Show that there is a scalar c such that $T = cS$.

Let $\{Sx_i : i \in I\}$ be a (Hamel) basis for the range of S. Then $i_1, i_2 \in I, i_1 \neq i_2 \Rightarrow T(x_{i_1} + x_{i_2}) = T(x_{i_1}) + T(x_{i_2})$ which gives $cS(x_{i_1} + x_{i_2}) = c_{x_{i_1}} Sx_{i_1} + c_{x_{i_2}} Sx_{i_2}$ for some c. The linear independence of Sx_{i_1} and Sx_{i_2} implies $c_{x_{i_1}} = c_{x_{i_2}} = c$. Thus, c_{x_i} is indpenedent of $i \in I$. In other words $Tx_i = \lambda Sx_i, i \in I$. Now let $x \in V$. Then Sx can be wriiten in the form $\sum_{j=1}^{N} \alpha_j Sx_{i_j}$. Let $y = x - \sum_{j=1}^{N} \alpha_j x_{i_j}$. Then $Sy = 0$ which implies $Ty = 0$. Thus $Tx = \sum_{j=1}^{N} \alpha_j Tx_{i_j} = \sum_{j=1}^{N} \lambda \alpha_j Sx_{i_j} = \lambda Sx$.

Problem 195

Let $f : (a, b) \to \mathbb{R}$ satisfy the following conditions:

f is 1-1, $\liminf_{y \to x+} f(y) \geq f(x)$, $\limsup_{y \to x-} f(y) \leq f(x)$ for all $x \in (a, b)$ and $\limsup_{y \to b-} f(y) = \inf_{a<x<b} f(x)$. Show that f is strictly decreasing and continuous. Give an example to show that the last condition cannot be dropped. The same conlcusion holds if $\liminf_{y \to a+} f(y) = \sup_{a<x<b} f(x)$.

The counter-example: $a = 0, b = \infty, f(x) = x$ for $0 < x < \frac{1}{2}$ and for $2 < x < \infty, f(x) = \frac{5}{2} - x$ for $\frac{1}{2} < x \leq 2$.

Now the proof: suppose, if possible there exist x_1, x_2 such that $x_1 < x_2$ and $f(x_1) < f(x_2)$. Since f is 1-1 we may increase x_1 to ensure that we also have $f(x_1) \neq \inf_{a<x<b} f(x)$. Let $A = \{x \in (x_2, b) : f(t) > f(x_1) \text{ on } (x_2, x)\}$. Since $\liminf_{y \to x_2+} f(y) \geq f(x_2) > f(x_1)$ it follows that $A \neq \emptyset$. Suppose $\alpha \equiv \sup A < b$. either $f(\alpha) < f(x_1)$ or $f(\alpha) > f(x_1)$. In the first case $\limsup_{y \to \alpha-} f(y) \leq f(\alpha) < f(x_1)$ so $f(y) < f(x_1)$ for all $y \in (\alpha - \delta, \alpha)$ for some $\delta > 0$; But then $(\alpha - \delta, \alpha) \cap A = \emptyset$ contradicting the the fact that $\alpha = \sup A$. In the second case $\liminf_{y \to \alpha+} f(y) \geq f(\alpha) > f(x_1)$ so $f(y) > f(x_1)$ for all $y \in (\alpha, \alpha + \eta)$ for some $\eta > 0$. There is a sequence $\{\alpha_n\} \subset A$ increasing to α and $f(y) > f(x_1)$ on (x_2, α_n) for each n. So $f(y) > f(x_1)$ on (x_2, α). We are assuming that $f(\alpha) > f(x_1)$ and we have proved that $f(y) > f(x_1)$ for all $y \in (\alpha, \alpha + \eta)$. Putting these together we see that $f(y) > f(x_1)$ for all $y \in (x_2, \alpha + \eta)$ contradicting the fact that $\alpha = \sup A$. We have now proved that $\alpha = b$. By hypothesis $\limsup_{y \to \alpha-} f(y) = \limsup_{y \to b-} f(y) = \inf_{a<x<b} f(x)$. However, $f(y) > f(x_1) > \inf_{a<x<b} f(x)$ on (x_2, α). This contradiction

shows that f is (strictly) decreasing. The hypothesis that $\liminf_{y \to x+} f(y) \geq f(x)$, $\limsup_{y \to x-} f(y) \leq f(x)$ for all $x \in (a, b)$ now shows that f is continuous. The last part is proved in a similar fashion by looking at $B = \{x : x < x_1$ and $f(t) < f(x_2)$ for $x < t < x_1\}$.

Problem 196

Compute $\max\{\min\{|x_i - x_j| : i \neq j\} : x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, \|x\| \leq 1\}$.

The answer is $\sqrt{\frac{12}{n(n^2-1)}}$. Let $x_i = a+bi, 1 \leq i \leq n$. Then $\|x\|^2 = \sum (a+bi)^2$. Maximize over $a \in \mathbb{R}$ to see that $\|x\| \leq 1$ if $|b| = \sqrt{\frac{12}{n(n^2-1)}}$. This shows the desired maximum is at least $\sqrt{\frac{12}{n(n^2-1)}}$. We now prove that it is at most $\sqrt{\frac{12}{n(n^2-1)}}$. The maximum is attained at some vector $(a_1, a_2, ..., a_n)$ and there is a permutation π of $\{1, 2, ..., n\}$ such that $b_1 \leq b_2 \leq ... \leq b_n$ where $b_i = a_{\pi(i)}, 1 \leq i \leq n$. For $j > i$ we have $b_j - b_i = (b_{i+1} - b_i) + (b_i - b_{i-1}) + ... + (b_j - b_{j-1}) \geq (j - i)\min\{b_j - b_i : j > i\} = (j - i)\min\{|a_j - a_i| : j \neq i\}$
$= (j - i)\max\{\min\{|x_i - x_j| : i \neq j\} : x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, \|x\| \leq 1\}$. It suffices to show that $\frac{b_j - b_i}{j - i} \leq \sqrt{\frac{12}{n(n^2-1)}}$ for some i and j with $i < j$. For this it suffices to show that $\sum_{i \neq j} (b_j - b_i)^2 \leq \frac{12}{n(n^2-1)} \sum_{i,j} (j - i)^2$. The exact value of the right side of this inequality is $\frac{12}{n(n^2-1)}[2n\frac{n(n+1)(2n+1)}{6} - 2\{\frac{n(n+1)}{2}\}^2] = 2n$. The left side is $\sum_{i,j} (a_j - a_i)^2 = 2n \sum a_i^2 - 2(\sum_j a_j)^2 \leq 2n$.

Problem 197

Let $n \in \{2, 3, 4, 5, 6, 7\}$. Does there exist an $n-$ times continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)f'(x)...f^{(n)}(x) < 0$ for all $x \in \mathbb{R}$?.

The function $f(x) = e^{-x}$ satisfies the desired inequality for $n = 2, 5$ and 6. The function $f(x) = -e^{-x}$ satisfies the desired inequality for $n = 4$. We prove that the inequality cannot hold for all x when $n = 3$ or $n = 7$; in fact the same is true when $n \equiv 3 (\bmod 4)$. As a first step we show that there is no twice continuously differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x)g''(x) < 0$ for all x. Indeed, if such function exists then g and g'' have no zeros and have constant signs on $\mathbb{R}$. Changing g to $-g$ if necessary we may suppose $g(x) < 0$ for all x and $g''(x) > 0$ for all x. g is then strictly convex. It is the upper envelop of its tangent lines, i.e. it has the form $g(x) = \sup\{a_i x + b_i : i \in I\}$. We have $a_i x + b_i < 0$ for all real x which implies $a_i = 0$ for all i. Thus g is a constant and $g'' \equiv 0$, a contradiction. Now suppose $n \equiv 3 (\bmod 4)$ and there exist an $n-$ times continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)f'(x)...f^{(n)}(x) < 0$ for all $x \in \mathbb{R}$. Then each of the functions $f^{(j)}, 0 \leq j \leq n$ have constant signs on $\mathbb{R}$. By what we just proved $f^{(j)}$ and $f^{(j+2)}$ have the same sign for $0 \leq j \leq n-2$. It follows that $f'(x)f^{(3)}(x)...f^{(4m+3)}(x)$ and $f(x)f^{(2)}(x)...f^{(4m+2)}(x)$ (where $n = 3+4m$) are both > 0 provided $f' > 0$. Multiplying thes two we see that $f(x)f'(x)...f^{(n)}(x) > 0$. Note that the desired inequality doesn't change if we replace f by $-f$. So there is no loss of generality in assuming that $f' > 0$ and the proof is complete when $n \equiv 3(\operatorname{mod} 4)$. What happens for other values of n? Note that if $\frac{n(n+1)}{2}$ is odd then $f(x) = e^{-x}$ serves as an example. if $4|n$ then $f(x) = -e^{-x}$ serves as an example. $n \equiv 2(\operatorname{mod} 4)$ or $n \equiv 1(\operatorname{mod} 4) \Rightarrow \frac{n(n+1)}{2}$ is odd.

Problem 198

If $x_1, x_2, ..., x_n \in \mathbb{R}^2, x_1 + x_2 + ... + x_n = 0, n \geq 2$ and $\|x_i\| \leq 1 \; \forall i$ show that $\|x_i + x_j\| \leq 1$ for some i and j.

W.l.o.g $x_i \neq 0$ for all i. By a rotation we may assume that x_1 is on the positive $x-$axis. If each x_j is on the $x-$axis then one of them, say x_j must be on the negative $x-$axis and $\|x_1 + x_j\| \leq 1$. In the contrary case at least one x_j is on the (open) upper-half plane. [Otherwise $\operatorname{Im}[x_1 + x_2 + ... + x_n] \leq 0$ which implies that all the vectors are on the $x-$axis]. Let x_2 be the one in the upper-half plane with maximum angle with the positive $x-$axis. If $\theta_2 \geq \frac{2\pi}{3}$ then $\|x_2 + x_1\|^2 = \|x_2\|^2 + \|x_1\|^2 + 2\|x_1\|\|x_3\|\cos(\theta_2 - 0) \leq \|x_2\|^2 + \|x_1\|^2 - \|x_1\|\|x_2\|$ $\leq \|x_1\|^2 \leq 1$ if $\|x_2\| \leq \|x_1\|$ and a similar argument holds if $\|x_1\| \leq \|x_2\|$. Assume now that $\theta_2 < \frac{2\pi}{3}$.

There must be another x_i, call it x_3 such that $\theta_3 > \theta_2$ where θ_j is the angle made by x_j with the positive $x-$ axis. To see this note that $\sum\limits_{j=1}^{n} \|x_j\| e^{i(\theta_j - \theta_2)} = 0$ and $\theta_j \leq \theta_2 \; \forall j$ would imply $\sin(\theta_j - \theta_2) = 0$ and $\theta_j - \theta_2 \in \{0, \pi, -\pi\}$ for all j forcing all the $x_j's$ to be on the $x-$axis. Thus $0 = \theta_1 < \theta_2 < \frac{2\pi}{3}$ and $\theta_2 < \theta_3$. But θ_2 is the largest of the angles θ_j corresponding to the $x_j's$ in the upper-half plane. it follows that $\theta_3 \geq \pi$. Now consider the cases $\theta_3 \geq \frac{4\pi}{3}$ and $\theta_3 < \frac{4\pi}{3}$. In the first case $\cos(\theta_3 - \theta_2) \leq -\frac{1}{2}$ and in the second case $\cos(\theta_3 - \theta_1) \leq -\frac{1}{2}$. In these cases we get (respectively) $\|x_2 + x_3\| \leq 1$ and $\|x_3 + x_1\| \leq 1$. This completes the proof.

Problem 199

Let $x_1, x_2.x_3, x_{4,}x_5, x_6$ be vectors in $\Delta \equiv \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ whose sum is 0. Show there are three vectors among these whose sum belongs to Δ.

By previous problem the sum of two of these is in Δ. Let $x_1 + x_2 \in \Delta$. Consider the 5 vectors $x_1 + x_2, x_3, x_{4,}x_5, x_6$. By previous problem again the sum of two of these is in Δ. If the sum $x_1 + x_2 + x_j, 3 \leq j \leq 6$ we are done. Suppose two of $x_3, x_{4,}x_5, x_6$ have a sum in Δ. Take these as x_3 and x_4. For simplicity we write x_{ij} for $x_i + x_j$. Thus x_{12} and x_{34} both belong to Δ. Once again we apply previous problem to $\{x_{12}, x_{34}, x_5, x_6\}$ and note that the only case where a proof is required is when $x_{56} \in \Delta$. Among the vectors x_{12}, x_{34}, x_{56} there must be two such that the angle between them does not exceed $\frac{2\pi}{3}$. [This is true for any three vectors!]. W.l.o.g assume that these are x_{12} and x_{34}. Rotate the vectors so that the positive x-axis is bisects the angle between these two and rename them, if necessary so that x_{12} is in the lower half plane. Let β_{ij} be the 'signed' angle between x_{ij} and the positive $x-$axis with $-\pi \leq \beta_{ij} < \pi$. Then $\beta_{12} = -\beta, \beta_{34} = \beta$ with $2\beta \leq \frac{2\pi}{3}$. Note that x_{12} is in the fourth quadrant and x_{34} is in the first quadrant. At least one of x_5, x_6 must be in the left half plane since the sum of all $x_i^{\prime}s$ is 0. Assume that x_5 is in the left half plane. If x_5 is in the upper half plane then the angle between x_{12} and x_5 is the smaller of the numbers $\alpha+\beta$ and $2\pi-\alpha-\beta$ where α is the angle made by x_5 with the positive $x-$axis. This angle is at least $\frac{2\pi}{3}$ provided $\beta > \pi/6$. In this case $\|x_1 + x_2 + x_5\| \leq 1$. Similarly, if x_5 is in the lower half plane then $\|x_3 + x_4 + x_5\| \leq 1$ provided $\beta > \pi/6$. To finish the proof we consider the case $\beta \leq \pi/6$. Here $< x_{12}, x_{34} > > 0$. We have $\|x_{12} + x_5\|^2 + \|x_{34} + x_5\|^2 + \|x_{12} + x_6\|^2 + \|x_{34} + x_6\|^2 = 2[\|x_5\|^2 + \|x_6\|^2 + \|x_{12}\|^2 + \|x_{34}\|^2 + < x_5 + x_6, x_{12} + x_{34} >]$

$= 2[\|x_5\|^2 + \|x_6\|^2 - 2 < x_{12}, x_{34} >] \leq 4$ which implies that one of the four terms is ≤ 1.

Problem 200

Show that there is no expanding continuous map from $\mathbb{R}^3$ to $\mathbb{R}^2$. [$f : \mathbb{R}^3 \to \mathbb{R}^2$ is expanding if $\|f(x) - f(y)\| \geq \|x - y\|$ for all x, y].

We prove that if (X, d) and (Y, ρ) are metric spaces and if there is an expand map from X to Y then the Huasdorff dimension of X does not exceed the Hausdorf dimension of Y. This would show that there is no expanding map from $\mathbb{R}^m$ to $\mathbb{R}^n$ if $m > n$ because the Hausdorff dimension of $\mathbb{R}^n$ is n. [Recall definition of Hausdorff dimension: let $A \subset X$ and consider $H(A, a) = \lim\limits_{\epsilon \to 0} \inf\{\sum\limits_n \{(diam(U_n)\}^a : A \subset \bigcup\limits_n U_n, U_n$ is open in $X, diam(U_n) \leq \epsilon \ \forall n\}$ where $a > 0$. Then $\inf\{a > 0 : H(A, a) = 0\}$ is the Hausdorf dimension of A]. Since Hausdorff dimension of $f(X)$ does not exceed that of Y we may assume $Y = f(X)$. Thus f is bijective. Let $g = f^{-1}$. Then g is a contraction: $\|g(x) - g(y)\| \leq \|x - y\|$ for all x, y. It follows that $H(g(E), a) \leq H(E, a)$ for all $a > 0$. Hence the dimension of $g(E)$ does not exceed the dimension of E. In particular the dimension of X does not exceed the dimension of Y.

Problem 201

Let $f : [0, 1] \to [0, 1]$ be continuous with $\int\limits_0^1 f(x)dx = 0$. Show that $a^2 f(a) =$

$\int_0^a (x+x^2)f(x)dx$ for some $a \in (0,1)$.

Let $g(x) = x^2 f(x) - \int_0^x (y^2+y)f(y)dy$. We have to show that this continuous function vanishes somewhere in $(0,1)$.

We prove that there exist $a, b \in [0,1]$ such that $g(a) < 0$ and $g(b) > 0$. This would finish the proof. Now, $g(x) \leq x^2 f(x) - m\int_0^x (y^2+y)dy = x^2 f(x) - m(\frac{x^3}{3} + \frac{x^2}{2}) = x^2 m - m(\frac{x^3}{3} + \frac{x^2}{2}) < 0$ if $f(x) = m \equiv \min\{f(t) : 0 \leq t \leq 1\}$. [We used the fact that $m < 0$ which follows from $\int_0^1 f(x)dx = 0$]. Similarly, $g(x) \geq x^2 f(x) - M\int_0^x (y^2+y)dy = x^2 M - M\int_0^x (y^2+y)dy = x^2 M - M(\frac{x^3}{3} + \frac{x^2}{2}) > 0$ if $f(x) = M \equiv \max\{f(t) : 0 \leq t \leq 1\}$. We have finished the proof. Note that the hypothesis $\int_0^1 f(x)dx = 0$ can be replaced by the weaker hypothesis $M > 0$ and $m < 0$.

Problem 202

Let X be a normed linear space over $\mathbb{R}$ and $T : X \to X$ satisfy $T(x+T(y)) = Tx + y$ for all $x, y \in X$. Prove that if $\sup\{\|Tx\| : \|x\| = 1\} < \infty$ then T is a continuous linear map with $T^2 = I$.

We have $T(T(0)) = T0 = T(-T0+T0) = T(-T0)+T0$. Hence $T(-T0) = 0$. If $Ty = 0$ then $Tx = Tx+y$ so $y = 0$. Taking $y = -T0$ we see that $T0 = 0$. Hence $T(0+Ty) = T0+y = y$. In other words, $T^2 = I$. Now $T(x+y) = T(x+T^2y) = Tx+Ty$. Thus T is additive. Hence $T(ax+by) = aT(x)+bT(y)$ if $a, b \in \mathbb{Q}$. Let $M = \sup\{\|Tx\| : \|x = 1\|\}$. Let $x \neq 0$ and u be a vector linearly independent of x. Consider $\left\|x + r\frac{ax+bu}{\|ax+bu\|}\right\|$ for $(a,b) \in \mathbb{R}^2\backslash\{(0,0\}$. Taking $b = 0$ and $a = \pm 1$ we see that $\left\|x \pm r\frac{x}{\|x\|}\right\| = \|x\| \pm r$ belong to the range of this continuous function of (a,b). If $0 < r < \|x\|$ and α is a rational number in $(\|x\| - r, \|x\| + r)$ then there exists $(a,b) \in \mathbb{R}^2\backslash\{(0,0\}$ such that $\left\|x + r\frac{ax+bu}{\|ax+bu\|}\right\| = \alpha$. [This follows from connectedness of $\mathbb{R}^2\backslash\{(0,0\}]$. Note that $\|v\| \in \mathbb{Q} \Rightarrow \|Tv\| = \|v\| \left\|T\frac{v}{\|v\|}\right\| \leq M\|v\|$. Thus taking $v = x + r\frac{ax+bu}{\|ax+bu\|}$ we get $\left\|Tx + rT(\frac{ax+bu}{\|ax+bu\|})\right\| \leq M\alpha$. By triangle inequality this yields $\|Tx\| \leq M\alpha + |r| M < M\alpha + \|x\| M \leq (\|x\| +$

$r)M+\|x\|\,M$. Letting $r\to 0$ through positive rationals we get $\|Tx\|\leq 2M\,\|x\|$. The rest is obvious.

Problem 203

Let $f:[0,1]\to\mathbb{R}$ be continuous and non-constant. Let m and M be the minimum and maximum of f on $[0,1]$. If $\int\limits_0^1 f(x)dx=0$ show that $\left|\int\limits_0^1 xf(x)dx\right|\leq\frac{-mM}{2(M-m)}$.

Let $F(x)=mI_{[0,M]}+MI_{(M,\alpha]}$ where $M-m=\alpha$. [Note that $m<0<M$]. We may suppose (by multiplying f by a positive number and changing f to $-f$ if necessary) that $\alpha=1$ and $\int\limits_0^1 xf(x)dx>0$. We have $\int\limits_0^1 xF(x)dx=\int\limits_0^M xF(x)dx+\int\limits_M^1 xF(x)dx=m\frac{M^2}{2}+M\frac{1-M^2}{2}=\frac{-mM}{2}=\frac{-mM}{2(M-m)}$. It suffices to show that $\int\limits_0^1 xf(x)dx\leq\int\limits_0^1 xF(x)dx$. Since $\int\limits_0^1 F=0$ we have $0=M\int\limits_0^1(f(x)-F(x))dx=M\int\limits_0^M(f(x)-F(x))dx+M\int\limits_M^1(f(x)-F(x))dx$

$$\geq\int\limits_0^M x(f(x)-F(x))dx+\int\limits_M^1 x(f(x)-F(x))dx=\int\limits_0^1 x(f(x)-F(x))dx.$$

Problem 204

Show that any function $f:\mathbb{R}\to\mathbb{R}$ is the composition of two Lebesgue measurable functions.

Let $h(\sum\frac{a_n}{2^n})=(\sum\frac{2a_n}{3^n}),g(\sum\frac{2a_n}{3^n})=f(\sum\frac{a_n}{2^n})$. Make g linear in each of the intervals removed in construction of Cantor's ternery set.

Problem 205

Say that a function $f:\mathbb{R}\to\mathbb{R}$ is Cesaro continuous if $c-\lim a_n=a\Rightarrow c-\lim f(a_n)=f(a)$. [$c-\lim a_n=a$ means $\frac{a_1+a_2+\ldots+a_n}{n}\to a$]. Find all Cesaro continuous functions $\mathbb{R}\to\mathbb{R}$.

Considering sequences of the type $\{x,y,x,y,\ldots\}$ we see that $f(\frac{x+y}{2})=\frac{f(x)+f(y)}{2}$. Also, $\{x:f(x)\leq a\}$ is closed for each a. Thus, f is measurable and hence it must be affine.

Problem 206

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. if $\int fg" \geq 0$ for every non-negative C^2 function g with compact support show that f is convex.

Fix ϵ, Δ and approximate f by a polynomial on $[-\Delta, \Delta]$ to within ϵ. Use integration by parts to conclude that $\int p'" g \geq 0$; conclude that p is convex.

Problem 207

Let μ be a real measure on the Borel $\sigma-$algebra of $\mathbb{R}$ such that $\mu(I) = 0$ for every interval of length 1 or $\sqrt{2}$. Show that μ is the zero measure.

$\mu(I)$ is zero of I is an interval whose length is a positive integer or a positive integral multiple of $\sqrt{2}$. Hence it is zero if $m(I) = n - m\sqrt{2}$ or $m(I) = m\sqrt{2} - n$ where n and m are positive integers. But $\{n + m\sqrt{2} : n, m \text{ integers}\}$ is dense. Hence there are arbitarily small positive numbers δ such that $\mu(I) = 0$ for intervals of length δ. It follows that $\mu\{x\} = 0$ for all x and that $\mu(I) = 0$ for any interval I.

Problem 208

Say that functions f and $g : \mathbb{R} \to \mathbb{R}$ are similar if there is a function $h : \mathbb{R} \to \mathbb{R}$ which is bijective and $f = h^{-1} \circ g \circ h$. Prove that sin and cos are not similar. Find all numbers a, b such that x^2 is similar to $x^2 + ax + b$.

If x^2 is similar to $x^2 + ax + b$ then there is a bijection h of $\mathbb{R}$ such that $h(x^2 + ax + b) = [h(x)]^2 \forall x$. Let $h(c) = 0$. Then $c^2 + ac + b = c$ since h maps both these points to 0. Also $x^2 + ax + b = c$ has at most one root since h is 0 at these roots. This equation has c as a root becuase h is 0 at bith sides of this equation. Hence the equation has equal roots. Therefore, $a^2 = 4(b - c)$. We get $a^2 = -4(c^2 + ac)$ which means $(a + 2c)^2 = 0$. Thus $a^2 = 4(b - c) = 4(b + a/2) = 4b + 2a$. Note that if this last condition holds then a bijection h satisfying $h(x^2 + ax + b) = [h(x)]^2 \forall x$ exists: take $h(x) = x + a/2$. Thus x^2 is similar to $x^2 + ax + b$ if and only if $a^2 = 4b + 2a$. What about $\sin x$ and $\cos x$?

Problem 209

If f and g are continuous functions : $\mathbb{R} \to \mathbb{R}$ which are periodic with period 1 show that $\int_0^1 f(x)g(ny)dy \to \int_0^1 f(x)dy \int_0^1 g(x)dy$. [This is called Fejer's Theorem].

By a simple change of variable this can be converted to the case where the period is 2π and the the range of integrals is from 0 to 2π. We can appriximate f and g by trigonometric polynomilas (e.g. cesaro averages of their Fourier series) so it is enough to consider the case when $f(x) = e^{ikx}$ and $g(x) = e^{imx}$. In this case direct evaluation gives the result.

Problem 210

Find $\lim \frac{\int_0^1 x^n f(x)dx}{\int_0^1 x^n g(x)dx}$ for continuous functions f and g on $[0,1]$ with $g > 0$ on $[0,1]$.

Make a change of variable. Ans.: $\frac{f(1)}{g(1)}$.

Problem 211

Suppose $\{f_n\}$ is a decreasing sequence of non-negative continuous functions on $[0,1]$ such that whenever f is continuous and $f_n \geq f \geq 0$ we have $f(x) = 0 \forall x$. Can we conclude that $\int_0^1 f_n(x)dx \to 0$?

No! Let U be an open set containing every rational number, with measure $1/2$. [Enough to construct and open set V containing rationals with measure not exceeding $1/2$; we can take U to be $V \cup (0,x)$ for a suitable x. For on $[0,1]$ we can take the union of sufficlently small intervals around rationals]. Let $f_n(x) = [1 - d(x,U^c)]^n$. Then $\int_0^1 f_n(x)dx \geq \int_{U^c} 1dx = 1/2 \forall n$. However, if f is as in the statement of the problem then $f(x) = 0$ whenever $d(x,U^c) > 0$ i.e. whenever $x \in U$. In particular $f = 0$ on Q and continuity forces f to be identically 0.

Problem 212

Prove that $\frac{a_n+b_n}{a_n+c_n} \to 1$ if $c_n \neq 0, a_n + c_n \neq 0 \forall n, -1$ is not a limit point of $\{\frac{a_n}{c_n}\}$ and $\frac{b_n}{c_n} \to 1$. Give an example to show that the condition that -1 is not a limit point of $\{\frac{a_n}{c_n}\}$ cannot be dropped. [If a_n, b_n, c_n are all > 0 then the condition can be dropped and the only hypothesis is $\frac{b_n}{c_n} \to 1$!]

For the counter-example take $a_n = \frac{1}{n^4} - \frac{1}{n}, b_n = \frac{1}{n^2} + \frac{1}{n}, c_n = \frac{1}{n}$.

Problem 213

If $\{a_n\}$ is a sequence in $\mathbb{R}\backslash\{0\}$ show that there is a subsequence $\{a_{n_k}\}$ such that $\{\frac{a_{n_{k+1}}}{a_{n_k}}\}$ converges to $0, 1$ or ∞.

First choose a subsequence that converges to -∞ or ∞ or a finite limit (which may be 0 or non-zero).

Problem 214 [See also Problem 86 above]
Let (X, d) be a metric space. Show that the following are equivalent:
a) Every continuous map $f : X \to \mathbb{R}$ is uniformly continuous
b) The distance between any two disjoint closed in X is positive.

a) implies b): Suppose A and B are disjoint closed sets with $d(A, B) = 0$. There is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. By a) f is uniformly continuous. By Problem 86 above $d(f(A), f(B)) = 0$. This gives the contradiction $|1 - 0| = 0$.

b) implies a): Suppose $f : X \to \mathbb{R}$ is continuous but not uniformly continuous. By Problem 86 there exists sets A and B such that $d(A, B) = 0$ but $d(f(A), f(B)) > 0$. We calim that the closed sets $\bar{A}$ and $\bar{B}$ are disjoint. If $x \in \bar{A} \cap \bar{B}$ then there exist sequences $\{a_n\}, \{b_n\}$ contained in A and B respectively both converging to x. Now $|f(a_n) - f(b_n)| \geq d(f(A), f(B)) > 0$ for each n which contradicts the fact that $f(a_n)$ and $f(b_n)$ both tend to $f(x)$. Thus $\bar{A}$ and $\bar{B}$ are disjoint. By b) $d(\bar{A}, \bar{B}) > 0$. However $d(\bar{A}, \bar{B}) \leq d(A, B) = 0$.

Problem 215 [See also Problem 86 and Problem 214]

Let (X, d) be a metric space which has at most finitely many isolated points. If every continuous function $f : X \to \mathbb{R}$ is uniformly continuous show that X is compact.

Remark: the example $X = \mathbb{N}$ shows that the assumtion on isolated points cannot be dropped]

Remark: if every real continuous function on a subset A of a metric space X is uniformly continuous then the subset is necessarily closed: if $x \in \bar{A}\backslash A$ then $y \to \frac{1}{d(x,y)}$ is a continuous function on A which is not uniformly continuous. [If $\{x_n\} \subset A$ and $x_n \to x$ then $d(x_n, x_m) \to 0$ but $\{\frac{1}{d(x,x_n)}\}$ is not Cauchy because it is not bounded]

Suppose X has a sequence $\{x_n\}$ with no convergent subsequence. Choose positive numbers $\delta_n, n = 1, 2, ...$ such that $B(x_n, \delta_n)$ does not contain any x_m

with $m \neq n$ and $\delta_n < \frac{1}{n}$ $\forall n$. Since x_n is not an isolated point for n sufficiently large (say, for $n \geq N$) we can find distinct points $x_{n,m}, m = 1, 2, ...$ in $B(x_n, \delta_n) \backslash \{x_n\}$ converging to x_n as $m \to \infty$ (for $n = N, N+1, ...$). We may suppose $d(x_{n,m}, x_n) < \frac{1}{n}$ $\forall n, m$. Let $A = \{x_{n,n} : n \geq N\}$ and $B = \{x_n : n \geq N\}$. Claim: A is closed. Suppose $x_{n_j,n_j} \to y$. Then $d(y, x_{n_j}) \leq d(y, x_{n_j,n_j}) + d(x_{n_j,n_j}, x_{n_j}) < d(y, x_{n_j,2m_j}) + \delta_{n_j} \to 0$. Thus $x_{n_j} \to y$. Since $\{x_n\}$ has no convergent subsequence it follows that $\{n_j\}$ is eventually constant and hence $y = \lim\limits_j x_{n_j,n_j}$ belongs to A. Clearly B is also closed. Further, $A \cap B = \emptyset$. Let $f : X \to [0, 1]$ be a continuous function such that $f(A) = \{0\}$ and $f(B) = \{1\}$. By hypothesis f is uniformly continuous. Note that $d(A, B) \leq d(x_n, x_{n,n}) < \frac{1}{n}$ $\forall n$. Thus $d(A, B) = 0$ and, by problem 86, $d(f(A), f(B)) = 0$. This contradicts the fact that $f(A) = \{0\}$ and $f(B) = \{1\}$. This contradiction shows that X is compact.

Problem 216

Let f be a fucntion from $\mathbb{R}$ to $\mathbb{R}$. If the restriction of f to $\mathbb{Q} \cup \{x\}$ is continuous for each irrational number x show that f is continuous.

Let $x \in \mathbb{R}, \epsilon > 0$ and consider $g^{-1}[f(x) - \epsilon, f(x) + \epsilon]$ where g is the restriction of f to $\mathbb{Q} \cup \{x\}$. This set is open in $\mathbb{Q} \cup \{x\}$ and it contains x. Hence there exists $\delta > 0$ such that if q is any rational number in $(x - \delta, x + \delta)$ then $|f(q) - f(x)| \leq \epsilon$. Combined with the hypothesis this shows that $y \in (x - \delta, x + \delta)$ implies $|f(y) - f(x)| \leq \epsilon$.

Problem 217

Let A be a closed subset of a metric space (X, d) and $f : A \to [1, 2]$ be a continuous function. Let $F(x) = f(x)$ if $x \in A$ and $F(x) = \frac{1}{d(x,A)} \inf\{f(y)d(x, y) : y \in A\}$ if $x \notin A$. Show that F is a continuous extension of f to X.

Let $\{x_n\} \to x$. If $x \notin A$ then $x_n \notin A$ for all n sufficiently large and $\frac{1}{d(x_n,A)} \inf\{f(y)d(x_n, y) : y \in A\} \to \frac{1}{d(x,A)} \inf\{f(y)d(x, y) : y \in A\}$ as $n \to \infty$. Indeed, $d(x_n, A) \to d(x, A) > 0$ and $f(y)d(x_n, y) \leq f(y)d(x, y) + f(y)d(x_n, x) \leq f(y)d(x, y) + 2d(x_n, x)$. Taking infimum over $y \in A$ and using a similar inequality in the reverse direction we get $F(x_n) \to F(x)$. Now let $x \in A$. We may split $\{x_n\}$ into two parts, one contained in A and the other in A^c, and, using continuity of f we mat reduce the proof to the case $x_n \notin A$ for any n. We have to show that

$\beta_n \equiv \frac{1}{d(x_n,A)} \inf\{f(y)d(x_n, y) : y \in A\} \to f(x)$. We first prove that $\limsup\limits_n \beta_n \leq f(x)$. We can find $y_n \in A$ with $d(x_n, A)(1 + \frac{1}{n}) > d(x_n, y_n)$ and $\beta_n \leq \frac{1}{d(x_n,A)} f(y_n)d(x_n, y_n) < (1 + \frac{1}{n})f(y_n)$. Also note that $d(x_n, A) \to d(x, A) = 0$ so $y_n \to x$. This gives $\limsup\limits_n \beta_n \leq f(x)$. Now $\inf\{f(y)d(x_n, y) : y \in A\} + \frac{1}{n} > f(y_n)d(x_n, y_n)$ for some $y_n \in A$. Thus $\beta_n > \frac{f(y_n)d(x_n,y_n)}{d(x_n,A)}$. In particular $2 \geq f(x) \geq \limsup\limits_n \beta_n \geq \limsup\limits_n \frac{d(x_n,y_n)}{d(x_n,A)}$ which implies that $d(x_n, y_n) \to 0$

and hence that $y_n \to x$. Thus, if $\epsilon > 0$ is given then $\beta_n > \frac{f(y_n)d(x_n,y_n)}{d(x_n,A)} > \beta_n > \frac{(f(x)-\epsilon)d(x_n,y_n)}{d(x_n,A)} \geq (f(x)-\epsilon)$ for n sufficiently large proving that $\liminf\limits_n \beta_n \geq f(x)$. The proof is complete.

Remark: the range $[1,2]$ can be replaced by any $[a,b]$ with $a < b$. It can also be replaced by any open interval or the whole line $\mathbb{R}$. Here is a proof when the range is $(-1,1)$: Let G be a continuous extension of f to X with values in $[-1,1]$. Let $C = G^{-1}\{-1,1\}$. Then A and C are disjoint closed sets and hence there is a function $\phi : X \to [0,1]$ such that $\phi(A) = \{0\}$ and $\phi(C) = \{1\}$. Let $F = (1-\phi)G$. Then $|F(x)| \leq 1$ for all x and equality can hold only when $|G(x)| = 1$ and $\phi(x) = 0$. But there is no x satisfying there properties and F is the desired extension.

Problem 218

Show that $[0,1]$ cannot be expressed as a countable union of (more than one) non-degenerate closed intervals. Show that the same is true of $(0,1)$.

[Problem 229 below contains a stronger result. Obviously, 'countable ' can be dropped]

Suppose $[0,1] = \bigcup\limits_n [a_n,b_n]$ with $\not{a}_n < b_n$ $\forall n$ and $[a_n,b_m] \cap [a_n,b_m] = \emptyset$ for $n \neq m$. Note that any collection of non-degenerate intervals is countable since such intervals contain at least one rational number. Let $U = \bigcup\limits_n (a_n,b_n)$ and $D = [0,1] \backslash U$. Then $D \subset \{a_n : n \geq 1\} \cup \{b_n : n \geq 1\}$. Since any perfect set in $\mathbb{R}$ is uncountable [Theorem 6.65 of Real and Abstract Analysis by Hewitt and Stromberg] we can complete the proof by showing that the closed set D is perfect. Suppose x is a isolated point of D. Suppose $x = a_k$ for some k. We can choose $\delta > 0$ such that $\delta < b_k - x$ and $(x-\delta, x+\delta) \cap D = \{x\}$. The point $x - \delta/2$ is in some $[a_n,b_n]$. If $b_n \geq x+\delta$ the we see that $x+\delta/2 \in [a_k,b_k] \cap [a_n,b_n]$ which forces k and n to be equal. But $x - \delta/2 < x = a_k = a_n$ contradicting the fact that $x - \delta/2$ is in some $[a_n,b_n]$. Thus $b_n < x+\delta$ and $b_n \in (x-\delta, x+\delta)$ [Indeed, $x - \delta < x - \delta/2 \leq b_n, x+\delta$]. But then $b_n \in (x-\delta, x+\delta) \cap D = \{x\}$. Thus $x = a_k = b_n$. If $n = k$ this contradicts the fac that $[a_k,b_k]$ is non-degenerate and if $n \neq m$ this contradicts the fact that $[a_k,b_k] \cap [a_n,b_n] = \emptyset$. This finishes the proof when x is a left end point of one the intervals $[a_n,b_n], n = 1,2,...$ and a similar argument holds when it is a right end point.

Now suppose an open interval (a,b) is a disjoint union of non-degenerate closed intervals. Then, using the fact that $[a-1,b+1] = [a-1,a] \cup (a,b) \cup [b,b+1]$ we can express $[a-1,b+1]$ in a similar way which is a contradiction.

Problem 219

If every real continuous function on a topological space is bounded then every real continuous function on it attains its supremum (and infimum).

If $M = \sup\{f(x) : x \in X\}$ then $\frac{1}{M-f}$ is continuous but not bounded.

Problem 220

Show that $X = [0,1]^{[0,1]}$ with the product topology is separable.

Let $k \in \mathbb{N}$ and $J_1, J_2, ..., J_k$ be k disjoint closed intervals with rational end points contained in $[0,1]$. Let $r_1, r_2, ..., r_k \in \mathbb{Q}$. Define $x \in X$ by $x(t) = r_j$ for all $t \in J_j, 1 \leq j \leq k$ and $x(t) = 0$ if t does not belong to any J_j. This defines a countable collection of points in X. Consider a basic open set $V = \{x : x(t_1) \in U_1, x(t_2) \in U_2, ..., x(t_k) \in U_k\}$ in X where k is a positive integer, $t_i's$ are distinct points of $[0,1]$ and $U_i's$ are open sets in $\mathbb{R}$. There exist disjoint closed intervals with rational end points containing $t_1, t_2, ..., t_k$. Pick rational numbers $r_1, r_2, ..., r_k$ in $U_1, U_2, ..., U_k$. Define $x \in X$ by $x(t) = r_j$ for all $t \in J_j, 1 \leq j \leq k$ and $x(t) = 0$ if t does not belong to any J_j. Then x belongs to the countable collection we just defined. Also this point lies in the basic open set V. This completes the proof.

Second proof: we prove that polynomials with rational coefficients are dense in X. Any basic open set $\{x : x(t_1) \in U_1, x(t_2) \in U_2, ..., x(t_k) \in U_k\}$ contains polynomial with rational coefficients: there is a continuous function in this set and this function can be approximated uniformly on $[0,1]$, hence on $\{t_1, t_2, ..., t_k\}$ by a polynomial with rational coefficients.

Problem 221

Show that $X = [0,1]^I$ with the product topology is not separable if the cardinality of I exceed the cardinality of power set of $\mathbb{N}$.

Let D be a dense subset of X. With each $i \in I$ associate the subset $D \cap p_i^{-1}(0,1)$ of D where $p_i : X \to [0,1]$ is the projection map $p_i(x) = x_i$. We claim that this map is one-to-one. If $i_1 \neq i_2$ then the dense set D must intersect the non-empty open set $p_{i_1}^{-1}(0,1) \cap p_{i_2}^{-1}(1,2)$. If $x \in p_{i_1}^{-1}(0,1) \cap p_{i_2}^{-1}(1,2) \cap D$ then $x \in p_{i_1}^{-1}(0,1) \cap D \backslash p_{i_2}^{-1}(0,1) \cap D$ and hence $p_{i_1}^{-1}(0,1) \cap D \neq p_{i_2}^{-1}(0,1) \cap D$. This proves our claim and shows that the cardinality of I does not exceed the cardinality of the power set of D.

Problem 222

Prove or disprove: if X is a compact metric space and $X = \bigcup\limits_{i \in I} U_i$ where each U_i has non-empty interior then X is covered by a finite number of $U_i's$.

False: let $X = [0,1]$. Let $U_n = [\frac{1}{n+1}, \frac{1}{n}], n = 2, 3, ...$ and $U_0 = [\frac{1}{2}, 1] \cup \{0\}$ then

$U_0 \cup U_1 \cup U_2 \cup ... \cup U_N$ does not contain $\frac{1}{N+2}$.

Problem 223

The one point compactification X of $\mathbb{N}$ is metrizable. Find a metric explicitly.

$d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|, d(n,\infty) = \frac{1}{n}$.

Problem 224

Find the cardinality of the Stone-Cech compactification of $\mathbb{N}$.

By definition of $\beta(\mathbb{N})$ the cardinality of $\beta(\mathbb{N})$ does not exceed that of $[0,1]^{([0,1]^{\mathbb{N}})}$ which is 2^c. Let D be a countable dense subset of $[0,1]^{[0,1]}$. [See problem 220 above]. Let $f : \mathbb{N} \to D$ be a bijection. Since f is continuous and $[0,1]^{[0,1]}$ is a compact Hausdorff space there is a continuous function $F : \beta(\mathbb{N}) \to [0,1]^{[0,1]}$ which extends f. Since $\mathbb{N}$ is dense in $\beta(\mathbb{N})$ and the range of F is compact it follows that F is onto. Hence the cardinality of $\beta(\mathbb{N})$ is at least equal to that of $[0,1]^{[0,1]}$ which is 2^c. Hence the cardinality of $\beta(\mathbb{N})$ is exactly 2^c.

Problem 225

Let (X,d) be a metric space and A be a subset of X such that $A \cap K$ is open in K for every compact set K. Show that A is open in X.

If x belongs to the closure of A^c then there is a sequence $\{x_n\}$ in A^c converging to x. Let $K = \{x, x_1, x_2, ...\}$. Then K is compact and $\{x_n\}$ is a sequence in the closed set $A^c \cap K$ converging to x. Hence $x \in A^c \cap K$. In particular $x \in A^c$. Thus A^c is closed.

[X can be any first countable Hausdorff space for this proof to be valid. A simpler argument shows that X can also be a locally compact Hausdorf space].

Problem 226

Let A be a G_δ in $\mathbb{R}$. Show that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous at all points of A and discontinuous at all points of A^c.

See Problem 260 below for another construction.

[Such a function can exists only if A is a G_δ].

Proof: let A be a G_δ in $\mathbb{R}$. Let $B = A^c$. Write B as a disjoint union of sets $E_1, E_2, ...$ such that $E_1 \cup E_2 \cup ... \cup E_n$ is a closed set C_n for each n. [Let $B = C_1 \cup C_2 \cup ...$ with each C_n closed and let $E_1 = C_1, E_n = C_n \backslash \{C_1 \cup C_2 \cup ... \cup C_{n-1}\}$ for $n \geq 2$]. Let $f(x) = \begin{cases} 0 \text{ if } x \in A \\ \frac{1}{n} \text{ if } x \in E_n \backslash (E_n^0) \\ \frac{1}{n+1} \text{ if } x \in E_n^0 \cap \mathbb{Q} \\ \frac{1}{n+2} \text{ if } x \in E_n^0 \backslash \mathbb{Q} \end{cases}$.

Note that if $x \notin A$ then $x \in E_n$ for a unique n and $x \in E_n \backslash (E_n^0)$ or $x \in E_n^0 \cap \mathbb{Q}$ or $x \in E_n^0 \backslash \mathbb{Q}$. Thus, f is a well defined function from $\mathbb{R}$ to $\mathbb{R}$. If $x \in A$ and $x_j \to x$ then $f(x) = 0$ and $f(x_j) \leq \frac{1}{n_j}$ if $x_j \in E_{n_j}$ (or $x_j \in A$). To show that f is continuous at x we only have to show that $n_j \to \infty$. If this is false then there is an integer k and a subsequence $\{j_l\}$ of $\{1, 2, ...\}$ such that

$x_{j_l} \in E_k \subset E_1 \cup E_2 \cup ... \cup E_k$ for each l. But the union is here is closed and hence $x \in E_1 \cup E_2 \cup ... \cup E_k \subset B$ which is a contradiction. Thus, f is continuous at each point of A.

Now let $x \in B$. Then $x \in E_n$ for some n. To prove that f is not continuous at x we prove that if V is any neighbourhood of x which intersects E_n then f is not a constant on V. [If f is continuous at x then for any $\epsilon > 0$ there is a neighbourhood V of x such that $|f(x) - f(y)| < \epsilon$ for all $y \in V$. We choose ϵ as follows: $f(x)$ (which belongs to $\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}\}$) has positive distance δ from $\{\frac{1}{j} : \frac{1}{j} \neq f(x)\}$. Let $0 < \epsilon < \delta$. If $y \in V$ and $f(y) \neq f(x)$ then $f(y)$ is $\frac{1}{j}$ for some $\frac{1}{j} \neq f(x)$ and so $\epsilon < |f(x) - f(y)|$, a contradiction. Thus f is constant on V]. Now, if $V \cap E_n^0 \neq \emptyset$ then $V \cap E_n^0 \cap \mathbb{Q} \neq \emptyset$ and $V \cap E_n^0 \backslash \mathbb{Q} \neq \emptyset$. Thus there is points y_1, y_2 of V such that $f(y_1) = \frac{1}{n+1}$ and $f(y_2) = \frac{1}{n+2}$. Thus f is not a constant on V. Now let $V \cap E_n^0 = \emptyset$. Let $y \in V \cap E_n \subset \partial E_n$. Then $V \cap (E_1 \cup E_2 \cup ... \cup E_{n-1})^c \neq \emptyset$. [In fact, $y \in V \cap (E_1 \cup E_2 \cup ... \cup E_{n-1})^c$]. Thus $V \cap (E_1 \cup E_2 \cup ... \cup E_{n-1})^c$ is a neighbourhood of y. Since $y \in \partial E_n$ it follows that $V \cap (E_1 \cup E_2 \cup ... \cup E_{n-1})^c \cap E_n^c \neq \emptyset$. Let $z \in V \cap (E_1 \cup E_2 \cup ... \cup E_{n-1})^c \cap E_n^c$. Then $f(z) = 0$ or $f(z) = \frac{1}{j}$ with $j \geq n+1$. In particular, $f(z) \leq \frac{1}{n+1}$. But $f(y) = \frac{1}{n}$ and hence $f(y) \neq f(z)$ completing the proof.

Remark: the result holds in any topological space which contains a set D such that D and D^c are both dense: we can replace $\mathbb{Q}$ by D in above proof. No space with an isolated point can have such a set and it is known that any first countable space without isolated points and any locally compact Hausdorf space without isolated points contains such a set. Ref: Sets of Points of Discontinuity by Richard Bolstein, Proceedings of AMS, Vol.38,No.1, 1973.

Problem 227

Show that there is a metric D on $\mathbb{R}$ such that $|x_n - x| \to 0$ implies $D(x_n, x) \to 0$ (equivalently every open set for D is open for the usual metric) and $(\mathbb{R}, D)$ is compact.

Consider the one-point compactification $X = (\mathbb{R}\backslash\{0\}) \cup \{\infty\}$ of $\mathbb{R}\backslash\{0\}$. [Neighbourhoods of ∞ are complements of compact subsets of $\mathbb{R}\backslash\{0\}$]. The map $f : \mathbb{R} \to X$ which is identity on $\mathbb{R}\backslash\{0\}$ and maps 0 to ∞ is a continuous bijection. [Indeed, if $x_n \to 0$ in $\mathbb{R}$ and U is a neighbourhood of ∞ then U^c is a compact subset of $\mathbb{R}\backslash\{0\}$ and hence $x_n \notin U^c$ for n sufficiently large]. Define $D(x, y) = d(f(x), f(y))$ where d is a metric for X. X is metrizable because $\mathbb{R}\backslash\{0\}$ is second countable and there is a countable local base at ∞ : $\{[-N, N]\backslash(-\frac{1}{N}, \frac{1}{N}) : N = 1, 2, ...\}$ is a countable base at ∞. Thus X is a second countable compact Hausdorf space and hence a compact metric space.

[The argument can be modified to prove a general result: any locally compact separable metric space has a smaller metrizable topology which makes it compact (cf. page 188, Exercise 113, of Wilansky). Also, if X is locally compact and Hausdorff then there exists a smaller topology on X which makes it a compact Hausdorff space: let $x \in X$ and $Y = (X\backslash\{x\}) \cup \{\infty\}$ be the one-point compactification of $X\backslash\{x\}$. Let $f : X \to Y$ be defined by $f(y) = y$ if $y \in X\backslash\{x\}$

and $f(x) = \infty$. If V is an open set containing ∞ then V^c is a compact subset of $X\backslash\{x\}$. We claim that this set is contained in the interior of another compact subset H of $X\backslash\{x\}$. This is an easy consequence of the fact that X is a locally compact Hausdorff space. Now $X\backslash H$ is a neighbourhood of x and $f(X\backslash H) \subset V$. Thus f is continuous. The weakest topology on X which makes f continuous is the required topology.]

Problem 228

Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous map such that $\frac{f(z)}{z} \to 1$ as $z \to \infty$. Show that f must vanish somewhere.

Let Δ be such that $\left|\frac{f(z)}{z} - 1\right| < 1$ for $|z| \geq \Delta$. Let N be a positive integer such that $N > \Delta$ and $N > \sup\{|f(z) - z| : |z| \leq \Delta\}$. If $|z| \leq N$ then either $|z| \leq \Delta$ or $\Delta < |z| \leq N$. In the first case $|f(z) - z| \leq N$ and in the second case $|f(z) - z| < |z| \leq N$. Thus $|z| \leq N$ implies $|f(z) - z| \leq N$. By Brower's Fixed Point Theorem the function $z - f(z)$ must have a fixed point.

Problem 229

Show that $[0,1]$ cannot be written as a countable disjoint union of two or more non-empty closed sets.

Conclude that a countable T_1 space contains no non-constant path.

The second part follows immediately from the first: if γ is a non-constant path then inverse images of singletons sets in the range of γ shows that $[0,1]$ can be written as a countable disjoint union of two or more non-empty closed sets.

Now suppose $[0,1] = \bigcup\limits_{n=1}^{\infty} C_n$ where each C_n is closed and non-empty and $C_n \cap C_m = \emptyset$ for $n \neq m$. [Note that $[0,1]$ cannot be written as a *finite* disjoint union of two or more non-empty closed sets. This is clear since these closed would also be open congtradicting connectedness of $[0,1]$]. By BCT at least one C_n has non-empty interior. Let $A = [0,1]\backslash \bigcup\limits_{n=1}^{\infty} C_n^0$. Clearly A is closed and $A = \bigcup\limits_{n=1}^{\infty} \partial C_n$. We claim that its interior is empty: if $[a,b] \subset A$ with $a < b$ then, by BCT again, there is an open interval $I \subset [a,b]$ and an integer k such that $I \subset \partial C_k$. But C_k is closed and hence ∂C_k has empty interior. This proves our claim. Now $A = \bigcup\limits_{n=1}^{\infty} \partial C_n$ and another application of BCT shows that there is an open interval (α, β) such that $(\alpha,\beta) \cap A$ is non-empty and contained in ∂C_j for some j. To complete the proof we show that there is at least one point in $(\alpha,\beta) \cap (A\backslash\partial C_j)$. Note that $(\alpha,\beta) \cap C_j^c$ is open and non-empty. [Let $x \in (\alpha,\beta) \cap A$. Then $x \in \partial C_j$

and (α,β) is a neighbourhood of x so this neighbourhood must intersect C_j^c]. If $(\alpha,\beta)\cap C_j^c$ is contained in A then A would have interior points contradicting the claim above. Thus $(\alpha,\beta)\cap C_j^c$ intersects A^c. Let $y\in(\alpha,\beta)\cap C_j^c\cap A^c$. Note that $y\in[0,1]=\bigcup\limits_{n=1}^{\infty}C_n$ and $y\in A^c$. By definition of A we see that $y\in C_m^0$ for some m. Observe that $(\alpha,\beta)\cap A^c\cap\partial C_m=\emptyset$ since $\partial C_m\subset A$. We are trying to prove that $(\alpha,\beta)\cap(A\backslash\partial C_j)$ is non-empty. If this set is empty then $(\alpha,\beta)\cap A\subset\partial C_j\subset C_j$ which implies that $(\alpha,\beta)\cap A\cap\partial C_m=\emptyset$. [Note that $y\in C_j^c\cap C_m^0$ and hence $m\neq j$]. Now $(\alpha,\beta)\cap A^c\cap\partial C_m=\emptyset$ and $(\alpha,\beta)\cap A\cap\partial C_m=\emptyset$ together yield $(\alpha,\beta)\cap\partial C_m=\emptyset$ and so (α,β) is the union of its intersection with C_m^0 and C_m^e (the exterior of C_m). By connectedness of this interval and the fact that $y\in(\alpha,\beta)\cap C_m^0$ we conclude that $(\alpha,\beta)\cap C_m^e=\emptyset$. But this means $(\alpha,\beta)\subset C_m$. This implies that $(\alpha,\beta)\subset C_m^0$ and hence (α,β) has no intersection with A. But $x\in(\alpha,\beta)\cap A$ and this finishes the proof.

We give another proof of the first part. This proof does not use BCT. suppose $[0,1]=\bigcup\limits_{n=1}^{\infty}C_n$ where each C_n is closed and non-empty and $C_n\cap C_m=\emptyset$ for $n\neq m$. If $U_2=\{x:d(x,C_2)<\frac{1}{2}d(C_1,C_2)$ then U_2 is open, $C_2\subset U_2$ and $\bar{U_2}\cap C_1=\emptyset$. Pick a point in C_2 and consider the component of that point in $\bar{U_2}$. Thus A_2 intersects C_2 but does not intersect C_1. A_2 is a closed interval. We claim that it contains at least one boundary point of U_2. Otherwise, $A_2\subset U_2$ and since A_2 is a compact interval it cannot be maximal: there is a larger open interval between A_2 and U_2. If x is a point of A_2 which is in ∂U_2 then $x\notin C_2$ (because C_2 is contained in the open set U_2). Thus, $A_2\backslash C_2\neq\emptyset$. Also $A_2\backslash C_2=\bigcup\limits_{n=3}^{\infty}(A_2\cap C_n)$because $A_2\cap C_1\subset\bar{U_2}\cap C_1=\emptyset$. Now $A_2\cap C_n\neq\emptyset$ for at least one $n>2$. We repeat the above argument for the interval A_2. By induction we get a sequence of compact intervals $A_2,A_3,...$ such that $A_{n+1}\subset A_n$ and $A_n\cap C_{n-1}=\emptyset$. There must be a point in the intersection of these intervals and that point cannot belong to any C_n. This contradiction completes the proof.

Problem 230

Let $f:(X,d)\to(Y,\rho)$ be a continuous and closed map. If $y\in Y$ show that $\partial f^{-1}\{y\}$ is compact.

Let $\{x_i\}$ be a sequence in $\partial f^{-1}\{y\}$. For each positive integer n the set $B(x_n,\frac{1}{n})\cap f^{-1}(B(y,\frac{1}{n}))$ is an open set containing x_n (because $\partial f^{-1}\{y\}\subset f^{-1}\{y\}$) and since $x_n\in\partial f^{-1}\{y\}$ there must be a point z_n in this open set which does not belong to $f^{-1}\{y\}$). Thus $z_n\in f^{-1}(B(y,\frac{1}{n})\backslash\{y\})$ and $d(z_n,x_n)<\frac{1}{n}$. Let $A=\{z_1,z_2,...\}$. Note that $f(z_n)\to y$ and hence y belongs to the closure of $f(A)$. However $y\notin f(A)$. Since f is continuous and closed the closure of $f(A)$ is same as $f(\bar{A})$. Thus A is not closed. [Because y belongs to the closure of $f(A)$

and $y \notin f(A)$]. Let $z_{n_j} \to u$ with $u \notin A$. Then $x_{n_j} \to u$ (because $d(z_n, x_n) < \frac{1}{n}$) and hence the given sequence $\{x_n\}$ has a convergent subsequence. [Note that $u = \lim x_{n_j}$ necessarily belongs to $\partial f^{-1}\{y\}$ because this set is closed].

Problem 231

Show that a metric space is compact if and only if it is complete under any equivalent metric.

If X is a compact metric space then it is so under any equivalent metric, so it is complete under any equivalent metric. Suppose now that X is a metric space which is complete under any equivalent metric.

Suppose X is not compact. Without loss of generality assume that the original metric d on X is such that $d(x,y) < 1$ for all $x, y \in X$. There exists a decreasing sequence of non-empty closed sets $\{C_n\}$ whose intersection is empty. Let $\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x,y)$ where $d_n(x,y) = |d(x,C_n) - d(y,C_n)| + \min\{d(x,C_n), d(y,C_n)\} d(x,y)$. We claim that ρ is a metric on X which is equivalent to d and that (X,ρ) is not complete. Note that $d_n(x,y) \leq 2$ for all $x, y \in X$. If x and $y \in C_k$ then x and $y \in C_n$ for $1 \leq n \leq k$ and hence $\rho(x,y) \leq \sum_{n=k+1}^{\infty} \frac{2}{2^n} = \frac{1}{2^k}$. Thus, the diameter of C_k in (X,ρ) does not exceed $\frac{1}{2^k}$. Once we prove that ρ is a metric equivalent to d it follows that ρ is not complete because $\{C_n\}$ is a decreasing sequence of non-empty closed sets whose intersection is empty.

Assuming (for the time being) that d_n satisfies triagle inequlaity it follows easily that ρ is a metric: if $\rho(x,y) = 0$ then $d(x,C_n) = d(y,C_n)$ for each n and $\min\{d(x,C_n), d(y,C_n)\} d(x,y) = 0$ for each n. If $d(x,y) \neq 0$ it follows that $d(x,C_n) = d(y,C_n) = 0$ for each n which implies that x and y belong to each C_n contradicting the hypothesis. Thus ρ is a metric. Also $\rho(x_j, x) \to 0$ as $j \to \infty$ implies $|d(x_j,C_n) - d(x,C_n)| \to 0$ and $\min\{d(x_j,C_n), d(x,C_n)\} d(x_j,x) \to 0$ as $j \to \infty$ for each n. There is at least one integer k such that $x \notin C_k$ and we conlude that $d(x_j,x) \to 0$. Conversely, suppose $d(x_j,x) \to 0$. Then $d_n(x_j,x) \to 0$ for each n and the series defining ρ is uniformly convergent, so $\rho(x_j,x) \to 0$. It remains only to show that d_n satisfies triangle inequality for each n . We have to show that $|d(x,C_n) - d(y,C_n)| + \min\{d(x,C_n), d(y,C_n)\} d(x,y)$

$\leq |d(x,C_n) - d(z,C_n)| + \min\{d(x,C_n), d(z,C_n)\} d(x,z) + |d(z,C_n) - d(y,C_n)| + \min\{d(z,C_n), d(y,C_n)\} d(z,y)$ for all x, y, z. Let $r_1 = d(x,C_n), r_2 = d(y,C_n), r_3 = d(z,C_n)$. We consider six cases depending on the way the numbers r_1, r_2, r_3 are ordered. It turns out that the proof is easy when r_1 or r_2 is the smalles of the three. We give the proof for the case $r_3 \leq r_1 \leq r_2$. (The case $r_3 \leq r_2 \leq r_1$ is similar). We have to show that

$r_2 - r_1 + r_1 d(x,y) \leq r_1 - r_3 + r_3 d(x,z) + r_2 - r_3 + r_3 d(z,y)$ which says $r_1 d(x,y) \leq 2r_1 - 2r_3 + r_3 d(x,z) + r_3 d(z,y)$. Since d satisfies trangle inequality it suffices to show that $r_1 d(x,z) + r_1 d(z,y) \leq 2r_1 - 2r_3 + r_3 d(x,z) + r_3 d(z,y)$.

But this last inequality is equivalent to $(r_1 - r_3)[d(x,z) + d(z,y)] \leq 2r_1 - 2r_3$. This is true because $d(x,z) + d(z,y) \leq 1 + 1 = 2$.

Problem 232

Any two countable dense subsets of $\mathbb{R}$ are homeomorphic.

The relative topology of a dense subset of $\mathbb{R}$ is same as the order topology. We prove that if $A = \{a_1, a_2, ...\}$ is a countable subset of $\mathbb{R}$ such that A has no largest or smallest element and between any two elements of A there is another element then there is an order isomorphism from A onto $T = \{\frac{j}{2^n} : j, n \in \mathbb{Z}\}$. This would prove that A and $\mathbb{Q}$ are both homeomorphic to T and hence A is homeomorphic to $\mathbb{Q}$. Define $f : A \to T$ as follows: (assume that $a_n's$ are distinct) let $f(a_1) = 0, f(a_{n_1}) = 1$ where n_1 is the least integer such that $a_{n_1} > a_1$. Let $f(a_{n_2}) = 2$ where n_2 is the least integer such that $a_{n_2} > a_{n_1}$, and so on. Let $f(a_{m_1}) = -1$ where m_1 is the least integer such that $a_{m_1} < a_1, f(a_{m_2}) = -1$ where m_2 is the least integer such that $a_{m_2} < a_{m_1}$, and so on. Let $f(a_{k_1}) = \frac{1}{2}$ where k_2 is the least integer such that $a_1 < a_{m_{k_2}} < a_{n_1}, f(a_{k_2}) = \frac{3}{2}$ where k_2 is the least integer such that $a_{n_1} < a_{k_2} < a_{n_2}$, and so on. We get a strictly increasing function from a subset of A to T. Note that if a_n is in the domain of this function so is a_{n+1} (why?). Thus, the domain is all of T.

Problem 233

Prove or disprove the following:
if (X, τ) is a topological space, A is a subspace of X and U, V are disjoint open sets in A then there are disjoint open sets U_1, V_1 in X such that $U = U_1 \cap A$ and $V = V_1 \cap A$. What happens if X is assumed to be metrizable?

Let $A = \{0, 1\}$ considered as a subspace of $\mathbb{R}$ with the co-finite topology. Then $\{0\}$ and $\{1\}$ are disjoint open sets which are intersections with A of disjoint open sets in $\mathbb{R}$ (since there are no disjoint non-empty open sets in $\mathbb{R}$!). The result is true if metrizability is added to the hypothesis. We prove a slighly more general result: if $\{U_i\}_{i \in I}$ is a collection of open sets in A then there exists a collection $\{V_i\}$ of open sets in X (indexed by I) such that whenever J is a finite subset of I and $\bigcap_{j \in J} U_j = \emptyset$ we also have $\bigcap_{j \in J} V_j = \emptyset$. We define $V_i's$ explicitly as follows: $V_i = \{x \in X : d(x, U_i) < d(x, A \backslash U_i)\}$. It is clear that V_i is open and its intersection with A is U_i. Suppose J is a finite subset of I and $\bigcap_{j \in J} U_j = \emptyset$. Suppose $y \in \bigcap_{j \in J} V_j$. Then $d(y, U_j) < d(y, A \backslash U_j)$ for each $j \in J$. For each j there exists $u_j \in U_j$ such that $d(y, u_j) < d(y, A \backslash U_j)$. Pick j such that $d(y, u_j) = \min\{d(y, u_l) : l \in J\}$. For some $l \neq j$ we have $u_j \in A \backslash U_l$ and hence $d(y, u_l) < d(y, A \backslash U_l) \leq d(y, u_j)$. This contradicts the choice of j.

[Any countable metric space withous isolated points is homeomorphic to $\mathbb{Q}$: Sierpinski, Fund. Math., 1920,11-16. Thus $\mathbb{Q} \cap [0,1]$ is homeomorphic to $\mathbb{Q}$. Direct proof?]

Problem 234 [See also Problem 121 above]

Let (X,d) be a compact metric space and $f : X \to \mathcal{P}(X) \backslash \{\emptyset\}$ be a map such that $d(a,b) \le d(x,y)$ whenever $x \in f(a)$ and $y \in f(b)$. [$\mathcal{P}(X)$ is the power set of X]. Show that $f(x)$ is a singleton set for each x and that f is an isometry from X onto itself [if we write $f(x)$ for the element of the singleton set $f(x)$].

Let $a, b \in X$ and define $\{a_n\}, \{b_n\}$ to be any two sequences in X such that $a_n \in f(a_{n-1})$ and $b_n \in f(b_{n-1})$ for each n with $a_0 = a, b_0 = b$. By compactness of X there exists $n_j \uparrow \infty$ such that $d(a_{n_j}, a_{n_k})$ and $d(a_{n_j}, a_{n_k}) \to 0$ as $j, k \to \infty$. Let $\epsilon > 0$ and choose m such that $d(a_{n_j}, a_{n_k}) < \epsilon$ and $d(a_{n_j}, a_{n_k}) < \epsilon$ for $j, k \ge m$. It follows by hypothesis that $d(a_{n_j - n_m}, a_0) \le d(a_{n_j}, a_{n_m}) < \epsilon$ and (similarly) $d(b_{n_j - n_m}, b_0) < \epsilon$ for $j \ge m$. Now $d(a,b) \le d(a_1, b_1) \le ... \le d(a_{n_j - n_m}, b_{n_j - n_m}) < d(a_{n_j - n_m}, a_0) + d(a_0, b_0) + d(b_{n_j - n_m}, b_0) < 2\epsilon + d(a,b)$. Since ϵ is arbitrary it follows that $d(a,b) = d(a_1, b_1)$. This equality holds whenever $a, b \in X$ and $a_1 \in f(a), b_1 \in f(b)$. Taking $a = b$we get $a_1 = b_1$ whenever $a_1, b_1 \in f(a)$. Thus f is single valued and $d(a,b) = d(f(a), f(b))$ for all $a, b \in X$. This implies that the range of f is closed. It is also dense: if $x \in X$ then $\{x, f(x), f(f(x)), ...\}$ has a convergent subsequence and hence this subsequence is Cauchy. It follows from the fact that f is an isometry that x can be approximated aribtrarily closely by points in the range of f. Thus the range is both closed and dense. It follows that f is an isometry of X onto itself.

Problem 235

Let X be a Hausdorff space K be a compact subset and U, V be open sets such that $K \subset U \cup V$. Show that there exist compact sets K_1 and K_2 such that $K_1 \subset U, K_2 \subset V$ and $K = K_1 \cup K_2$.

Let $A = K \backslash V, B = K \backslash U$. Since A and B are compact and disjoint there exist open sets S and T such that $A \subset S$ and $B \subset T$. Replacing S and T by their intersections with U and V respectively we may suppose $S \subset U$ and $T \subset V$. Let $K_2 = K \backslash S$ and $K_1 = K \backslash T$. Note that $K \backslash S \subset K \backslash A \subset V$. Similarly, $K \backslash T \subset U$. Also $K = K_1 \cup K_2$ because S and T are disjoint.

Problem 236

There exists a compact metric space X and a homeomorphism $f : X \to X$ (onto) such that f is not an isometry under any equivalent metric.

Let $X = \{z \in \mathbb{C} : |z| = 1\}$ and $f(e^{it}) = e^{4\pi it/(2\pi+t)}$ for $0 \leq t \leq 2\pi$. Then the $n-th$ iterate f_n of f is given by $f_n(t) = e^{i\frac{4\pi t}{2\pi+(2^{n-1}-1)t}}$. Thus $f_n(t) \to 1$ as $n \to \infty$ for each t. If f is an isometry for some metric d compatible with the usual topology of X then $0 = d(1,1) = \lim d(f_n(t), f_n(s)) = d(f(t), f(s))$ for any pair (t,s) which leads to the contradiction that f is a constant.

Problem 237

Let $\Omega = \mathbb{N}, \mathcal{F} =$ power set of $\mathbb{N}$ and $P\{n\} = \frac{1}{2^{n!}}$ for $n = 2, 3, ..., P\{1\} = 1 - \sum_{n=2}^{\infty} \frac{1}{2^{n!}}$. Show that there are no non-constant independent random variables on this probability space.

Let X and Y be independent random variables on $(\Omega, \mathcal{F}, P)$ and suppose they are both non-constant. Let E be a non-empty Borel set in $\mathbb{R}$ which does not contain $X(1)$ and F be a non-empty Borel set which

does not contain $Y(1)$. We prove that $P\{X^{-1}(E) \cap Y^{-1}(F)\} \neq P\{X^{-1}(E)\}P\{Y^{-1}(F)\}$. We have $P\{X^{-1}(E)\} = \sum_{X(n)\in E} \frac{1}{2^{n!}}, P\{Y^{-1}(F)\} = \sum_{Y(n)\in F} \frac{1}{2^{n!}}$ and $P\{X^{-1}(E) \cap Y^{-1}(F)\} = \sum_{X(n)\in E, Y(n)\in F} \frac{1}{2^{n!}}$. Let $A = \{n : X(n) \in E\}$ and $B = \{n : Y(n) \in F\}$. If $P\{X^{-1}(E) \cap Y^{-1}(F)\} = P\{X^{-1}(E)\}P\{Y^{-1}(F)\}$ then we have $\sum_{n\in A} \frac{1}{2^{n!}} \sum_{n\in B} \frac{1}{2^{n!}} = \sum_{n\in A\cap B} \frac{1}{2^{n!}}$. This gives $\sum_{n\in A, m\in B} \frac{1}{2^{n!+m!}} = \sum_{k\in A\cap B} \frac{1}{2^{k!}}$. We look at the two sides as expansions to base 2 of some number in $(0,1)$. We note that $n! + m! = k! + j!$ implies $(n,m) = (k,j)$ or $(n,m) = (j,k)$. To see this suppose n is the least of the integers n, m, k, j and divide both sides by $(n+1)!$. We get $\frac{1}{n+1} \in \mathbb{Z}$, a contradiction unless j or k equals n. If $k = n$ then we get $m! = j!$ so $m = j$. Thus in the sum $\sum_{n\in A, m\in B} \frac{1}{2^{n!+m!}}$ each term is repeated at most twice. If $k \in A \cap B$ we must have $\frac{1}{2^{k!}} = \frac{1}{2^{n!+m!}}$ or $\frac{1}{2^{k!}} = \frac{2}{2^{n!+m!}}$. Hence $n! + m! = k!$ or $n! + m! - 1 = (k!)$. We note that $n! + m!$ can never be a factorial, nor can $n! + m! - 1$ be a factorial since -1 is not divisible by 2! Thus $A \cap B$ is empty. This contradicts the equation $\sum_{n\in A, m\in B} \frac{1}{2^{n!+m!}} = \sum_{k\in A\cap B} \frac{1}{2^{k!}}$ and the proof is complete.

Problem 238

Let (X,d) be a metric space without isolated points. If every continuous function from X into $\mathbb{R}$ is uniformly continuous prove that X is compact.

Suppose X is not compact. Let $\{x_n\}$ be a sequence with no convergent subsequence. There exists a sequence $\{y_n\}$ such that $0 < d(x_n, y_n) < \frac{1}{n}$. The set $\{x_n : n \geq 1\} \cup \{y_n : n \geq 1\}$ has no limit points. Define $f(x_n) = n, f(y_n) = 2n, n = 1, 2, ...$ Extend f to a continuous function on X. The extended function is obviuosly not uniformly continuous.

Problem 239 [Probabilistic construction of a strictly increasing continuous singular function]

Let $\{X_n\}$ be i.i.d. random variables with $P\{X_n = 0\} = p = 1 - P\{X_n = 1\}$ where $0 < p < 1, p \neq \frac{1}{2}$. Let $X = \sum\limits_{n=1}^{\infty} \frac{X_n}{2^n}$ and $F(x) = P\{X \leq x\}$. We claim that F is continuous and strictly increasing on $[0, 1]$ with $F' = 0$ a.e. If $a_n \in \{0, 1\}$ for each n then $P\{X_n = a_n \forall n\} = 0$. Thus $P\{X = x\} = 0$ for each x. [Indeed, $X = x = \sum\limits_{n=1}^{\infty} \frac{a_n}{2^n}$ forces $\{X_n\}$ to take at most two values in $\{0,1\}^{\mathbb{N}}$). Thus F is continuous. Now $P\{\frac{j}{2^n} < X < \frac{j+1}{2^n}\} = P\{X_k = a_k, 1 \leq k \leq n\}$ where $a_k's$ are determined by $\frac{j}{2^n} = \sum\limits_{k=1}^{n} \frac{a_k}{2^k}$. Hence, F is strictly increasing. Since monotonic functions are differentiable a.e. it suffices to show that if $0 < x < 1$ and F is differentiable at x then $F'(x) = 0$. For each n there exists j_n such that $\frac{j_n}{2^n} < x \leq \frac{j_n+1}{2^n}$. Let $I_n = (\frac{j_n}{2^n}, \frac{j_n+1}{2^n}]$. Then $\frac{P\{X \in I_n\}}{2^{-n}} = \frac{F(\frac{j_n+1}{2^n}) - F(\frac{j_n}{2^n})}{2^{-n}} \to F'(x)$. If $F'(x) \neq 0$ this gives $\frac{P\{X \in I_{n+1}\}}{P\{X \in I_n\}} \to \frac{1}{2}$. It is easy to see that $P\{X \in I_n\}$ is of the type $p_{a_1} p_{a_2} ... p_{a_n}$ where $p_1 = p$ and $p_0 = 1 - p$. Thus $\frac{P\{X \in I_{n+1}\}}{P\{X \in I_n\}} \in \{0, 1\}$ for each n and hence it cannot converge to $\frac{1}{2}$.

Problem 240

Give a proof of DCT (Dominated Convergence Theorem) without using Monotone Convergence Theorem or Fatou's Lemma.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $\{f_n\}$ a sequence of meaurable functions converging a.e. to a measurable function f such that $|f_n| \leq g$ a.e. and let g be integrable. Let $A_n = \{x : |f_k(x) - f(x)| \geq \delta g(x)$ for some $k \geq n\}$ where $\delta > 0$. Then $A_n \cap \{x : g(x) > 0\} \downarrow A$ where A has measure 0. Now $\int |f_n - f| \, d\mu = \int_{A_n} |f_n - f| \, d\mu + \int_{A_n^c} |f_n - f| \, d\mu$ and $\int_{A_n^c} |f_n - f| \, d\mu \leq \delta \int g d\mu$. Also $\int_{A_n} |f_n - f| \, d\mu \leq \int\limits_{A_n \cap \{x : g(x) > 0\}} 2g d\mu$. It reamins only to show that $\int\limits_{B_n} g d\mu \to 0$ if $B_n \downarrow B$ and $\mu(B) = 0$. This is obviously true for a simple integrable function g and the general case follows from the fact that there is an integrable simple function h with $\int |g - h| \, d\mu$ as small as we need.

Problem 241

Let $\{a_n\} \subset \mathbb{R}^k, \|a_n\| \to \infty$ and $\inf\{\|a_n - a_m\| : n \neq m\} > 0$. Show that $\sum_{n=1}^{\infty} \frac{1}{\|a_n\|^{k+\delta}} < \infty$ for every $\delta > 0$ but $\sum_{n=1}^{\infty} \frac{1}{\|a_n\|^{k}}$ may be ∞.

Let $K_N = \{x \in \mathbb{R}^k : |x_i| \leq N \text{ for each } i\}$. Claim: if $A \subset K_N \backslash K_{N+1}$ and $\|a - b\| \geq r$ for any two distinct points of A then the cardinality of A does not exceed $c[\frac{2(2N+r)}{r}]^k - c[\frac{2(2N-2-r)}{r}]^k$ where $\frac{1}{c}$ is the Lebesgue measure of the ball with center 0 and radius 1. For this let $x^{(1)}, x^{(2)}, ..., x^{(m)} \in A$ and note that $B(x^{(j)}, \frac{r}{2}), 1 \leq j \leq m$ are disjoint and they are all contained in $\{x : |x_i| \leq N + \frac{r}{2}$ for each i and $|x_i| > N - 1 - \frac{r}{2}$ for some $i\}$. Taking Lebesgue measure we see that $m \leq c\frac{(2N+r)^k - (2N-2-r)^k}{(\frac{r}{2})^k}$. Now $\sum_{n=1}^{\infty} \frac{1}{\|a_n\|^{k+\delta}} = \sum_{N=1}^{\infty} \sum_{a_n \in K_N \backslash K_{N+1}} \frac{1}{\|a_n\|^{k+\delta}} \leq \sum_{N=1}^{\infty} c\frac{(2N+r)^k - (2N-2-r)^k}{(\frac{r}{2})^k} \frac{1}{N^{k+\delta}} < \infty$ because $(2N + r)^k - (2N - 2 - r)^k < (2r + 2)kt^{k-1} \leq (2r + 2)k(2N + r)^{k-1}$ for some t between $2N - 2 - r$ and $2N + r$. We now give an eaxmple to show that we cannot take $\delta = 0$. Let S_N be the set formed by the points $(-N + rj_1, -N + rj_2, ..., -N + rj_k)$ where $j_i's$ are positive integers not eaxceeding $\frac{2N}{r}$. The cardinality of $S_N \backslash S_{N=1}$ is at least equal to $(\frac{2N}{r})^k - (\frac{2N-2}{r})^k$. Arranging $\bigcup_N (S_N \backslash S_{N=1})$ in a sequence $\{a_n\}$ we see that $\sum_{n=1}^{\infty} \frac{1}{\|a_n\|^k} \geq \sum_{N=1}^{\infty} \frac{1}{(N-1)^k}\{(\frac{2N}{r})^k - (\frac{2N-2}{r})^k\} \geq \alpha \sum \frac{1}{N}$ for some positive constant α.

Problem 242

Let A be a connected subspace of a connected space X. If C is a connected component of A^c show that C^c is connected.

We first show that if S is a clopen subset of A^c then $A \cup S$ is connected. Suppose $A \cup S = U \cup V$ with U and V open disjoint and non-empty in $A \cup S$. Then $A \subset U$ or $A \subset V$. Suppose, for definiteness, $A \subset V$. Then U is a clopen subset of S. [$U \subset A \cup S$ and $U \subset V^c \subset A^c$]. Hence it is a clopen subset of A^c. U is also a clopen subset of $A \cup S$ and hence it is a clopen subset of $A^c \cup (A \cup S) = X$. [Indeed $V \subset A \cup S$ and $V^c \subset A^c$]. But X is connected and we have arrived at a contradiction. This proves that $A \cup S$ is connected. Now suppose $C^c = E \cup F$ where E and F are non-empty disjoint open sets in C^c. Since $C \subset A^c$ we have $A \subset C^c$. Thus A is a connected subset of $E \cup F$. It follows that $A \subset E$ or $A \subset F$. For definiteness, let $A \subset E$. Now $C \cup F$ is connected by the result just proved (with C in place of A and F in place of S). This connected set is contained in A^c (because $F \subset E^c \subset A^c$) and it contains C strictly, contradicting the fact that C is a component of A^c.

Problem 243

Consider the set $A = \{\sum_{n=0}^{\infty} a_n x^n : \sum_{n=0}^{\infty} |a_n| < \infty\}$ as a subset of $C[0,1]$. Is this set of first category?

Define $T : l_1 \to C[0,1]$ by $T\{a_n\} = \sum_{n=0}^{\infty} a_n x^n$. T is a continuous linear map with a dense range by Weierstrass Approximation Theorem. Also, T is injective (by basic facts on power series). If the range of T is of second category then the proof of open mapping theorem shows that it is an open map. This would imply that the range is complete, hence closed. But then T would be surjective but not every continuous real function on $[0,1]$ has a power series expansion. In fact $\sqrt{x}$ is not differentiable and hence it does not have a power series expansion.

Problem 244

Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{y \to x} |f(y)| = \infty$ for every rational number x?

No! Since $\mathbb{R} = \bigcup_{n=1}^{\infty} \{x : |f(x)| < n\}$ there exists N and $a < b$ such that (a,b) is contained in the closure of $\{x : |f(x)| < N\}$. If $y \in (a,b)$ then there exists a sequence $\{y_j\}$ converging to y such that $|f(y_j)| < N$ for each j. [If no such sequence exists then there exists $\delta > 0$ such that $|f(z)| \geq N$ for all $z \in (y-\delta, y+\delta) \backslash \{y\}$. But then no point of $(y-\delta, y+\delta) \backslash \{y\}$ can belong to in the closure of $\{x : |f(x)| < N\}$ which is a contradiction]. But if we take y to be a rational number in (a,b) we get a contradiction to the hypothesis.

Remarks: the result holds with $\mathbb{R}$ replaced by a complete metric space and $\mathbb{Q}$ replaced by a dense subset. In particular there is no function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{y \to x} |f(y)| = \infty$ for every irrational number x. We can also prove this by considering $\left|f(\sqrt{2}x)\right| + \left|\frac{1}{x}\right| I_{\mathbb{R} \backslash \{0\}}$

Problem 245

Let M be a closed linear space contained in $C[0,1] \cap C'[0,1]$ (where $C'[0,1]$ is the set of all continuously differentiable real functions on $[0,1]$). Show that M if finite dimensional.

Define $T : M \to C[0,1]$ b\y $T(f) = f'$. T is well defined and linear. We now show that T has closed graph: let $f_n \to f$ and $f'_n \to g$ in $C[0,1]$. Then f is differentiable and $f' = g$ as seen from the relation $f_n(t) = f_n(0) + \int_0^t f'_n(s)ds$.

By Closed Graph Theorem T is continuous. Hence $|f'(x)| \leq \|T\| \|f\|$ for all $f \in M$. By Arzela Ascoli Theorem it follows that the closed unit ball of M is compact. Hence M is finite dimensional.

Problem 246

Let $x_n \to x$ weakly in a Hilbert space H. What can we say about $\liminf \|x_n - x\|$ and $\limsup \|x_n - x\|$ other then the fact that $\liminf \leq \limsup$?

Nothing! We prove that given $0 \leq a \leq b < \infty$ we can find an example where $\liminf \|x_n - x\| = a$ and $\limsup \|x_n - x\| = b$. Let $x_n = a_n e_n$ in $H = l^2$ where $\{a_n\}$ is a bounded sequence. Then $x_n \to 0$ weakly. Also $\liminf \|x_n - 0\| = \liminf a_n$ and $\limsup \|x_n - 0\| = \limsup a_n$.

Problem 247

Let $\{C_n\}$ be a decreasing sequence of closed convex non-empty bounded sets in a Hilbert space H. show that $\bigcap\limits_{n=1}^{\infty} A_n$ is non-empty.

Proof: there exists $x_n \in C_n$ such that $\|x_n\| = \inf\{\|x\| : x \in C_n\}$. Note that $\inf\{\|x\| : x \in C_n\} \leq \inf\{\|x\| : x \in C_{n+1}\}$. Hence $\{\|x_n\|\}$ is an increasing sequence of real numbers.. It is also bounded because $x_n \in C_n \subset C_1$. Now $\|x_{n+m} - x_n\|^2 = 2\|x_{n+m}\|^2 + 2\|x_n\|^2 - \|x_{n+m} + x_n\|^2$and $\frac{x_{n+m}+x_n}{2} \in C_n$, so $\|x_{n+m} + x_n\| \geq 2\|x_n\|$. So $\|x_{n+m} - x_n\|^2 \leq 2\|x_{n+m}\|^2 + 2\|x_n\|^2 - 4\|x_n\|^2 = 2(\|x_{n+m}\|^2 - \|x_n\|^2) \to 0$. Let $x_n \to x$. Since $\{x_n, x_{n+1}, ...\}$ is contained in C_n it follows that $x \in C_n$ for each n.

Problem 248

If P and Q are projections on a Hilbert space show that $\|P - Q\| \leq 1$ and $\|P + Q\| \geq 1$ if $PQ = QP$ and $P \neq Q$.

Second part is trivial since $\|P - Q\| = \left\|P^2 - Q^2\right\| \leq \|P - Q\| \, \|P + Q\|$. To prove that first part let $R = I - P$ and $S = I - Q$. Then $P = PQ + PS$ and $Q = PQ + RQ$ so $P - Q = \{PQ + PS\} - \{PQ + RQ\} = PS - RQ$. Since the ranges of P and R are orthogonal we get $\|Px - Qx\|^2 = \|PSx\|^2 + \|RQx\|^2 \leq \|Sx\|^2 + \|Qx\|^2 = \|x\|^2$.

Problem 249 [Non-metrizability of pointwise convergence topology]

Let ρ be the metric on $X = C[0,1]$ defined by $\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}$ and τ be the topology on X with $\{f : |f(x_i) - f_0(x_i)| < \epsilon_i, 1 \leq i \leq n\}$ where $n \in \mathbb{N}, \epsilon_i' s > 0$ and $x_i' s \in [0,1]$ as basic neighbourhoods of f_0 for each $f_0 \in X$.

Consider the identity map I from (X, τ) to (X, ρ). Show that this map is sequentially continuous but not continuous and that τ is not metrizable.

Since pointwise convergence of a sequence of continuous functions implies convergence in measure it follows that I is sequentially continuous. If it is continuous then there exist $n \in \mathbb{N}, \epsilon_i' s > 0$ and $x_i' s \in [0,1]$ such that $\{f : |f(x_i) - f_0(x_i)| < \epsilon_i, 1 \leq i \leq n\} \subset \{f : \int \frac{|f|}{1+|f|} < \frac{1}{2}\}$. In particular $\int \frac{|f|}{1+|f|} < \frac{1}{2}$ whenever f is of the type $c(x - x_1)(x - x_2)...(x - x_n)$. We get a contradiction by letting $c \to \infty$.

Problem 250

Show that a homeomorphism from $\mathbb{Q}$ onto itself need not be monotonic and its inverse need not be continuous.

Let $a, b \in \mathbb{Q}^c$ with $a < b, a + b \in \mathbb{Q}$ and $f(x) = \begin{cases} x \text{ if } x \leq a \text{ or } x \geq b \\ a + b - x \text{ if } a < x < b \end{cases}$. [We may take $a = \sqrt{2}, b = 1 - a$].

Problem 251

Give an example of a strong metric (other than the discrete metric), i.e. a metric d on a set X such that $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all x, y, z. Show that for any strong metric open balls are closed and, given any two open balls either one is contained in the other or the two balls are disjoint.

Let p be a fixed prime. For any positive integer n let $\phi(n)$ be the largest positive integer k such that p^k divides n. Clearly, $\phi(nm) = \phi(n) + \phi(m)$. Let $\phi(\pm \frac{n}{m}) = \phi(n) - \phi(m)$. It is trivial to check that this is well defined on the set of all non-zero rational numbers and that $\phi(xy) = \phi(x) + \phi(y)$ for $x, y \in \mathbb{Q} \backslash \{0\}$. Define $d(x, y) = p^{-\phi(x-y)}$ if x and y are distinct rational numbers and 0 if $x = y \in \mathbb{Q}$. Claim: d is a strong metric. For this we have to show that $\phi(x - y) \geq \min\{\phi(x - z), \phi(z - y)\}$. Equivalently we have to show $\phi(x - y) \geq \min\{\phi(x), \phi(y)\}$ if $x, y \in \mathbb{Q} \backslash \{0\}$ and $x \neq y$. W.l.o.g. let $\phi(x) \geq \phi(y)$. In this case we have to show $\phi(x - y) \geq \phi(y)$ which is equivalent to $\phi(\frac{x}{y} - 1) \geq 0$. Here $z = \frac{x}{y}$ is in $\mathbb{Q} \backslash \{0, 1\}$ and $\phi(z) \geq 0$. Write z as $\frac{n}{m}$ $(n, m \in \mathbb{N}, n \neq m)$ .Since $p^{\phi(m)}$ divides both n and m (because $\phi(m) \leq \phi(n)$) we see that $p^{\phi(m)}$ divides $n - m$ too and so $\phi(n - m) \geq \phi(m)$ as required. Thus d is a string metric.

Now let d be strong metric on a set X. We claim that $d(x, y) \neq d(y, z) \Rightarrow d(x, z) = \max\{d(x, y), d(y, z)\}$. Suppose first that $d(x, y) < d(y, z)$. We have to show that $d(x, z) \geq d(y, z)$. [The reverse inequality follows from triangle inequality]. But $d(y, z) \leq \max\{d(y, x), d(x, z)\} = d(x, z)$ [because if this last maximum is $d(y, x)$ then we would have $d(y, z) \leq \max\{d(y, x), d(x, z)\} = d(x, y)$ a contradiction]. Now consider the case $d(y, z) < d(x, y)$. To show $d(x, z) \geq \max\{d(x, y), d(y, z)\} \equiv d(x, y)$. But $d(x, y) \leq \max\{d(y, z), d(x, z)\} = d(x, z)$

[because if the maximum here is $d(y,z)$ then we would have $d(x,y) \leq d(y,z)$, a contradiction]. This proves the claim in all cases.

Let $U = B(x,r)$. If $y \in U^c$ and $d(z,y) < r$ we claim that $z \in U^c$ (proving that U^c is open and hence U is closed). If $z \in U$ then $d(z,x) < r$. Now $d(y,x) \leq \max\{d(y,z), d(x,z)\} < r$, a contradictionsince $y \in U^c$. Thus, every open ball in X is closed. Suppose now that $z \in B(x,r) \cap B(y,s)$. Since $z \in B(x,r)$ we have $B(z,r) \subset B(x,r) : d(u,z) < r \Rightarrow d(u,x) \leq \max\{d(u,z), d(x,z)\} < r$. We can interchange x and z to conclude that $B(x,r) \subset B(z,r)$. Thus, $B(z,r) = B(x,r)$ Similarly we get $B(z,s) = B(y,s)$. Thus $B(x,r) \subset B(y,s)$ or $B(y,s) \subset B(x,r)$ according as $r \leq s$ or $s \leq r$.

Problem 252

Prove or disprove the following:

for any subsequence $\{n_k\}$ of $\{1, 2, ...\}$ the sequence $\{\frac{1}{N}\sum\limits_{j=1}^{N} e^{in_j x}\}$ converges to 0 a.e. w.r.t Lebesgue measure on $\mathbb{R}$.

True. Let $f_N(x) = \frac{1}{N}\sum\limits_{j=1}^{N} e^{in_j x}$. Then $\frac{1}{2\pi}\sum\limits_{j=1}^{\infty}\int\limits_0^{2\pi} \left|f_{j^2}(x)\right|^2 dx = \sum\limits_{j=1}^{\infty}\frac{1}{j^2} < \infty$, so $\sum\limits_{j=1}^{\infty}\left|f_{j^2}(x)\right|^2 < \infty$ a.e. (on $[0, 2\pi]$ hence on $\mathbb{R}$). In particular $f_{j^2} \to 0$ a.e.. Now, for $N^2 < k \leq (N+1)^2$ we have $f_k(x) - f_{N^2}(x) = \frac{1}{k}\sum\limits_{j=1}^{k} e^{in_j x} - \frac{1}{N^2}\sum\limits_{j=1}^{N^2} e^{in_j x} = (\sum\limits_{j=1}^{N^2} e^{in_j x})(\frac{1}{k} - \frac{1}{N^2}) + \frac{1}{k}\sum\limits_{j=N^2+1}^{k} e^{in_j x}$. Since $\sum\limits_{j=1}^{\infty}(\frac{1}{k} - \frac{1}{N^2}) \leq \sum\limits_{j=1}^{\infty}(\frac{1}{N^2} - \frac{1}{(N+1)^2}) < \infty$ and $\frac{k-N^2}{k} \leq \frac{(N+1)^2-N^2}{N^2} \to 0$ we are done.

Problem 253

Let $f \in L^1([a,b])$. Show that f is Riemann integrable in the following modified sense:

given $\epsilon > 0$ there is a function $\delta : [a,b] \to (0,\infty)$ such that for any partition $\{x_i\}$ of $[a,b]$ and any choice if points ξ_i in $[x_{i-1}, x_i]$ satisfying the condition $x_i - x_{i-1} < \delta(\xi_i)$ we have $\left|\sum\limits_{j=1}^{N} f(\xi_i)[x_i - x_{i-1}] - \int f dm\right| < \epsilon$. [$m$ denotes Lebesgue measure].

There exists $\eta > 0$ such that $m(A) < \eta \Rightarrow \int\limits_A |f|\, dm < \epsilon/3$. Let $r = \frac{\epsilon}{3(\eta+b-a)}$. Let $E_j = f^{-1}((j-1)r, jr])$ for each $j \in \mathbb{Z}$. Choose open sets U_j such that $E_j \subset U_j$ and $m(U_j \backslash E_j) < \frac{\eta}{3(2^{|j|})(1+|j|)}$. Any point x in $[a,b]$ belongs to E_j for a unique j and we define $U_x = U_i$. We define $\delta(x)$ as $d(x, U_x^c)$. Consider any partition $\{x_i\}$ of $[a,b]$ and any choice if points ξ_i in $[x_{i-1}, x_i]$ satisfying the condition $x_i - x_{i-1} < \delta(\xi_i)$. Note that $[x_{i-1}, x_i] \subset U_{\xi_i} = U_{j_i}$ where j_i is such that $\xi_i \in E_{j_i}$. [If $y \in [x_{i-1}, x_i]$ then $|y - \xi_i| \leq x_i - x_{i-1} < \delta(\xi_i) = d(\xi_i, U_{j_i}^c)$ which implies $y \in U_{j_i}$]. Now $\left|\sum_{j=1}^{N} f(\xi_i)[x_i - x_{i-1}] - \int f dm\right| \leq \sum_{j=1}^{N} \int\limits_{[x_{i-1},x_i]} |f(\xi_i) - f(x)|\, dx \leq$ $S_1 + S_2 + S_3$ where $S_1 = \sum_{j=1}^{N} \int\limits_{[x_{i-1},x_i] \cap E_{j_i}} |f(\xi_i) - f(x)|\, dx$,

$S_2 = \sum_{j=1}^{N} \int\limits_{[x_{i-1},x_i] \backslash E_{j_i}} |f(\xi_i)|\, dx$ and $S_3 = \sum_{j=1}^{N} \int\limits_{[x_{i-1},x_i] \backslash E_{j_i}} |f(x)|\, dx$. Since $|f(\xi_i) - f(x)| \leq r$ for all $x \in [x_{i-1}, x_i] \cap E_{j_i}$ (because x and ξ_i both belong to E_{j_i}) we get $|S_1| \leq r[b-a] < \frac{\epsilon}{3}$. Next we note that $|S_2| = \sum_{m=-\infty}^{\infty} \sum_{\{i:j_i=m\}} \int\limits_{[x_{i-1},x_i] \backslash E_m} |f(\xi_i)|\, dx$

$\leq (|m|+1)r \sum_{m=-\infty}^{\infty} m(U_m \backslash E_m) < (|m|+1)r \sum_{m=-\infty}^{\infty} \frac{\eta}{3(2^{|m|})(1+|m|)}$ (since the sets $[x_{i-1}, x_i] \backslash E_m$ as i varies over all indices with $j_i = m$ are disjoint and they are all contained in $U_m \backslash E_m$) so $|S_2| \leq r\eta < \frac{\epsilon}{3}$. [we have used the elementary fact that $(j-1)r < t \leq jr$ implies $|t| \leq (1+|j|)r$]. Finally we look at S_3. Let $A = \bigcup_{i=1}^{N} ([x_{i-1}, x_i] \backslash E_{j_i})$. Then $m(A) \leq \sum_{j=1}^{N} m([x_{i-1}, x_i] \backslash E_{j_i}) =$ $\sum_{m=-\infty}^{\infty} \sum_{\{i:j_i=m\}} m([x_{i-1}, x_i] \backslash E_{j_i}) = \sum_{m=-\infty}^{\infty} m(U_m \backslash E_m)$ (since the sets $[x_{i-1}, x_i] \backslash E_m$ as i varies over all indices with $j_i = m$ are disjoint and they are all contained in $U_m \backslash E_m$) and so $m(A) < \eta$. It follows from the definition of η that $\int\limits_A |f| < \epsilon/3$. It follows that $|S_3| < \epsilon/3$ and hence that $\left|\sum_{j=1}^{N} f(\xi_i)[x_i - x_{i-1}] - \int f dm\right| < \epsilon$.

Remark: if δ is a constant function then f is Riemann integrable (and hence continuous a.e.).

Problem 254

Is $(\mathbb{R}^2, \|\|_1)$ isometrically isomorphic to $(\mathbb{R}^2, \|\|_\infty)$?

Yes. The map $(x, y) \to (\frac{x+y}{2}, \frac{x-y}{2})$ is an isometric isomorphism of $(\mathbb{R}^2, \|\|_\infty)$ onto $(\mathbb{R}^2, \|\|_1)$ because $|x| \leq \left|\frac{x+y}{2}\right| + \left|\frac{x-y}{2}\right|, |y| \leq \left|\frac{x+y}{2}\right| + \left|\frac{x-y}{2}\right|$ and $\left|\frac{x+y}{2}\right| + \left|\frac{x-y}{2}\right| \in \{x, -x, y, -y\}$.

Problem 255

Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that there is a function $g : [0, b-a] \to \mathbb{R}$ such that g is continuous, monotonically increasing, $g(0) = 0$ and $|f(x) - f(y)| \leq g(|x - y|)$ for all $x, y \in [a, b]$.

Let $g(t) = \sup\{|f(x) - f(y)| : x, y \in [a, b]$ and $|x - y| \leq t\}$ for $0 < t \leq b - a$ and $g(0) = 0$. It is trivial to see that $|f(x) - f(y)| \leq g(|x - y|)$ for all $x, y \in [a, b]$ and that g is monotonically increasing and continous at 0. Now let $0 < t < b - a$. Suppose $\{t_n\} \downarrow t$. Let $\epsilon > 0$ and choose $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Let $|x - y| \leq t_n$. We can find z such that $|x - z| \leq t$ and $|z - y| \leq t_n - t$. [$z = x + \frac{t}{t_n}(y - x)$ will do]. Hence $|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| < g(t) + \epsilon$ if n is so large that $t_n - t < \delta$. Taking supremum over all pairs (x, y) such that $|x - y| \leq t_n$ we get $g(t) \leq g(t_n) < g(t) + \epsilon$ if n is sufficiently large. This proves that g is right continuous. Now let $t_n \uparrow t$. Let $|x - y| \leq t$. We can find z such that $|x - z| \leq t_n$ and $|z - y| \leq t - t_n$. [$z = x + \frac{t_n}{t}(y - x)$ will do] Hence $|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| < g(t) + \epsilon$ if n is so large that $t - t_n < \delta$. Taking supremum over all pairs (x, y) such that $|x - y| \leq t$ we get $g(t_n) \leq g(t) < g(t_n) + \epsilon$ if n is sufficiently large. This proves that g is left continuous.

Problem 256

Let A be a subset of a metric space X and $f : A \to \mathbb{R}$ be continuous. Show that there exists a G_δ set B and a continuous function $F : B \to \mathbb{R}$ such that $A \subset B$ and $F(x) = f(x)$ for all $x \in A$.

Let $B = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \{x \in \bar{A} : |f(y) - f(z)| < \frac{1}{k}$ whenever $y, z \in A$, $d(x, y) \leq \frac{1}{m}$ and $d(x, z) \leq \frac{1}{m}\}$. Suppose $|f(y) - f(z)| < \frac{1}{k}$ whenever $y, z \in A, d(x, y) \leq \frac{1}{m}$ and $d(x, z) \leq \frac{1}{m}$. If $d(x, u) \leq \frac{1}{2m}$ then, for $y, z \in A$ with $d(u, y) \leq \frac{1}{2m}$ and $d(u, z) \leq \frac{1}{2m}$ we have $d(x, y) \leq \frac{1}{m}$ and $d(x, z) \leq \frac{1}{m}$ and hence $|f(y) - f(z)| < \frac{1}{k}$. This proves that $\bigcup_{m=1}^{\infty} \{x \in \bar{A} : |f(y) - f(z)| < \frac{1}{k}$ whenever $y, z \in A, d(x, y) \leq \frac{1}{m}$ and $d(x, z) \leq \frac{1}{m}\}$ is open in $\bar{A}$ for each k. Hence B is a G_δ set. By continuity of f on A it follows that $A \subset B$. If $x \in B$ and $\{x_n\}$ is a sequence in A converging to x the sequence $\{f(x_n)\}$ is Cauchy in $\mathbb{R}$. Let $F(x)$ be the limit of this sequence.

It is clear that this number does not depend on the particular sequence $\{x_n\}$ and that F is a continuous extension of f.

Problem 257

Any union of non-degenerate intervals is a countable union of intervals, hence a Borel set.

Let $E = \bigcup_{\alpha \in A} I_\alpha$ where $\{I_\alpha\}_{\alpha \in A}$ is a collection of intervals of positive length. If $x \in E$ let J_x be the union of all the intervals I_α which contain x. Then J_x is convex, hence an interval. [Consider $ty + (1-t)z$ where $0 < t < 1, y \in I_{\alpha_1}$ and $z \in I_2$. $I_{\alpha_1} \cup I_{\alpha_2}$ is connected because $x \in I_{\alpha_1} \cap I_{\alpha_2}$. Hence $I_{\alpha_1} \cup I_{\alpha_2}$ is an interval and so it contains $ty + (1-t)z$]. It is clear that J_{x_1} and J_{x_2} are either equal or disjoint for any two points x_1, x_2 in E. Pick a rational from each of these intervals to get an injective map from $\{J_x : x \in E\}$ into $\mathbb{Q}$. It follows that $E = \bigcup_{x \in E} J_x$ is a countable union on intervals.

Problem 258

Find all compact subgroups of S^1 (under multiplication).

Let G be a compact subgroup of S^1. Let $H = \{x \in \mathbb{R} : e^{ix} \in G\}$. Then H is a subgroup of $(\mathbb{R}, +)$. Hence it is either dense or discrete. If it is dense then $G = S^1$: for any real number x there is a sequence $\{x_n\}$ in H converging to x and $e^{ix} = \lim e^{ix_n} \in G$ since G is closed. In the remaining case there exists a positive number a such that $H = \{na : n \in \mathbb{Z}\}$. We consider two cases: $\frac{a}{2\pi} \in \mathbb{Q}$ and $\frac{a}{2\pi} \in \mathbb{Q}^c$. In the first case let $\frac{a}{2\pi} = \frac{p}{q}$ where p and q are positive integers with no common factors. We have $G = \{e^{i2\pi np/q} : n \in \mathbb{Z}\} = \{e^{i2\pi np/q} : n \in \{0, 1, ..., q-1\}$. Thus, G is the group of $q-th$ roots of unity in this case. If $\frac{a}{2\pi} \in \mathbb{Q}^c$ we claim that $G = S^1$: the set $\{n + m\frac{a}{2\pi} : n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Given $x \in \mathbb{R}$ let $n_j + m_j \frac{a}{2\pi} \to \frac{x}{2\pi}$. Then $e^{iam_j} = e^{2\pi n_j + iam_j} \to e^{ix}$ and $e^{iam_j} \in G$ for each j proving that the closed set G is also dense in S^1.

Problem 259

Using Uniform Boundedness Principle show that there exists a continuous periodic function whose Fourier series at 0 does not converge.

[0 can be replaced by any other point. We prove below the existence of a continuous periodic function whose the partial sums of whose Fourier series at 0 form an unbounded sequence].

Let X denote the Banach space of all continuous complex functions f on $[-\pi, \pi]$ satisfying $f(-\pi) - f(\pi)$ with the supremum norm and let $S_N(f, x)$ denote the $N-th$ partial sum of the Fourier series of f at x. Define $T_N : X \to \mathbb{C}$ by

$T_N f = S_N(f, 0)$. If the Fourier series of every $f \in X$ at 0 converges then, by Uniform Boundedness Principle, $\sup\limits_N \|T_N\| < \infty$. We show that this is false. Let $\epsilon > 0, N \in \mathbb{N}$ and δ_N be a positive number such that $\int\limits_E |D_N(x)|\,dx < \epsilon$ if $m(E) < \delta_N$. The function $D_N(x) = \frac{\sin(N+\frac{1}{2})x}{\sin(\frac{1}{2}x)}$ has only a finite number of zeros on $[-\pi, \pi]$. Let U be the union of small intervals around these points so that (the intervals are disjoint and) $m(U) < \delta_N$. Let f be a real valued continuous function on $[-\pi, \pi]$ which is equal to $\frac{|D_N(x)|}{D_N(x)}$ on U^c and linear in each of the intervals that make up U. By the standard expression for $S_N(f, 0)$ in terms of D_N we have $T_N f = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f D_N$. Hence, $T_N f = \frac{1}{2\pi}\int\limits_{U^c} |D_N| + \frac{1}{2\pi}\int\limits_U f D_N = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |D_N| - \frac{1}{2\pi}\int\limits_U |D_N| + \frac{1}{2\pi}\int\limits_U f D_N$. Noting that $\left|\frac{1}{2\pi}\int\limits_U f D_N\right| \leq \frac{1}{2\pi}\int\limits_U |D_N| < \epsilon$. Thus $T_N f > \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |D_N| - 2\epsilon$. It is well known that $\{\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |D_N|\}$ is unbounded; in fact $\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |D_N| - \frac{4}{\pi^2}\log N$ is bounded. Hence the proof is complete. [See p. 154 of Fourier Series by Edwards for an explicit construction].

Problem 260

Let A be a G_δ subset of $\mathbb{R}$. Give a simple construction of a fucntion $f : \mathbb{R} \to \mathbb{R}$ which is continuous precisely at points of A.

See Problem 226 above for another construction.

Let $A = \bigcap\limits_{n=1}^{\infty} G_n$ with G_n open and $G_{n+1} \subset G_n$ for all n. Let $f_n = I_{C_n \backslash E_n}$ where $C_n = G_n^c$ and $E_n = \mathbb{Q} \cap C_n^0$. Let $f = \sum\limits_{n=1}^{\infty} \frac{1}{n!} f_n$. We claim that f has the desired properties. First let $x \in A$. Then $f_n(x) = 0$ for all n. In fact, for each n, f_n vanishes in a neighbourhood of x. Hence each f_n is continuous at x. By uniform convergence of the series defining f we see that f is also continuous at x. Now let $x \in A^c$. Let k be the least positive integer such that $x \in C_k$. If $x \in C_k^0$ then, in sufficiently small neighbourhoods of x, f_k take both the values 0 and 1 and so its oscillation at x is 1. We claim that the oscillation of f_j at x is 0 for each $j < k$: since $x \notin C_j$ it follows that points close to x are all in C_j^c and hence f_j vanishes at those points. Now $\omega(f,x) \geq \frac{1}{k!}\omega(f_k,x) - \sum_{j=k+1}^{\infty} \frac{1}{j!}$ since $\omega(f_j,.) \leq 1$ everywhere. Thus $\omega(f,x) \geq \frac{1}{k!} - \sum_{j=k+1}^{\infty} \frac{1}{j!} \geq \frac{1}{k!}[1 - \sum_{j=k+1}^{\infty} \frac{1}{(k+1)(k+2)...(j)}] > \frac{1}{k!}[1 - \sum_{j=k+1}^{\infty} \frac{1}{2^{j-k}}] = 0.$

Problem 261

If $f \in L^1(\mathbb{R})$ and $2f(t) = 3f(3t) + 3f(3t-1)$ a.e. show that $f = 0$ a.e.

Probability theory makes this quite simple. Let $\{X_n\}$ be i.i.d. random variable taking values 0 and 2 each with probability $\frac{1}{2}$ and $X = \sum_{n=1}^{\infty} \frac{X_n}{3^n}$. Then X takes values in the cantor set. Let μ be a probability measure induced by X. Then $\mu(C) = 1$ and hence $\mu \perp m$ where m is the Lebesgue measure. Note that $\int e^{itx} d\mu(x) = Ee^{itX} = \prod_{n=1}^{\infty} Ee^{it\frac{X_n}{3^n}} = \prod_{n=1}^{\infty} \frac{1+e^{i2t/3^n}}{2}$. From this we conclude that there is no L^1 function whose Fourier transform is $\prod_{n=1}^{\infty} \frac{1+e^{i2t/3^n}}{2}$. However, the given equation yields $2\hat{f}(t) = \hat{f}(t/3) + e^{it/3}\hat{f}(t/3)$ where $\hat{f}(t) = \int e^{itx} f(x)dx$.

Thus $\hat{f}(t) = \hat{f}(t/3)(\frac{1+e^{it/3}}{2})$. Iteration gives $\hat{f}(t) = \hat{f}(t/3^k)(\frac{1+e^{it/3}}{2})(\frac{1+e^{it/3^2}}{2})...(\frac{1+e^{it/3^k}}{2})$. Letting $k \to \infty$ we get $\hat{f}(t) = \{\prod_{n=1}^{\infty} \frac{1+e^{i2t/3^n}}{2}\}\hat{f}(0)$. This would lead to the contradiction that $\prod_{n=1}^{\infty} \frac{1+e^{i2t/3^n}}{2}$ is the Fourier transform of an L^1 function unless $\hat{f}(0)$ which implies $\hat{f}(t) = 0$ for all t and hence $f = 0$ a.e.

Problem 262

Let $f : (0,1) \to (0,\infty)$ be any function with $\lim_{x \to 0+} f(x) = \infty$. Show that there is a non-negative convex function g on $(0,1)$ such that $\lim_{x \to 0+} g(x) = \infty$ and $g(x) \leq f(x)$ for all $x \in (0,1)$.

Choose a sequence of positive numbers $\{\alpha_n\}$ such that $\alpha_{n+1} < \frac{\alpha_n}{2}, \inf\{f(x) : \alpha_{n+1} \leq x < \alpha_n\} > n$ and $\alpha_n \to 0$. Let $a_n = -[\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ... + \frac{1}{\alpha_n}], n = 1, 2, ...$ Let $g(x) = a_n x + n$ for $\alpha_{n+1} \leq x < \alpha_n$. We have $g(x) \geq a_n\alpha_n + n$ for $\alpha_{n+1} \leq x < \alpha_n$ and $a_n\alpha_n + n = n - \alpha_n[\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ... + \frac{1}{\alpha_n}] > n - [1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}}]$ which is positive for $n > 1$ and $\to \infty$ as $n \to \infty$ so $\lim_{x \to 0+} g(x) = \infty$ and g is positive on $(0,t)$ where $t = \alpha_2$ and $g(x) \leq n < \inf\{f(x) : \alpha_{n+1} \leq x < \alpha_n\} \leq f(x)$ for $\alpha_{n+1} \leq x < \alpha_n$ proving that $g \leq f$. On the open interval (α_{n+1}, α_n) we have

$g'(x) = a_n$. Since $\{a_n\}$ is decreasing we can conclude that g is convex if we can show that g is continuous at the points $\alpha_n, n = 1, 2, ...$ The right hand limit of g at α_n is $a_{n-1}\alpha_n + n - 1$ and the left hand limit is $a_n\alpha_n + n$. However $a_n\alpha_n + n = \alpha_n[a_n - a_{n-1}] + \alpha_n a_{n-1} + n = -1 + \alpha_n a_{n-1} + n = a_{n-1}\alpha_n + n - 1$. We have proved that there exists a positive convex function g on $(0, t)$ such that $\lim\limits_{x \to 0+} g(x) = \infty$ and $g(x) \leq f(x)$ for all $x \in (0, t)$. It is clear from this proof that if f is bounded below by a positive constant δ then (by applying above argument to $\frac{2}{\delta} f$) we can construct a positive convex function g on $(0, 1)$ which is bounded above by f such that $\lim\limits_{x \to 0+} g(x) = \infty$. For the general case let g_n be a positive convex function on $(0, 1)$ which is bounded above by $\max\{f, \frac{1}{n}\}$ and satisfies the condition $\lim\limits_{x \to 0+} g(x) = \infty$. Then $\limsup g_n$ is non-negative, convex, bounded above by f and $\limsup g_n(x)$ tends to ∞ as $x \to 0$.

Remark: the result becomes false if $(0, 1)$ is replaced by $(1, \infty)$ and the condition $\lim\limits_{x \to 0+} g(x) = \infty$ by $\lim\limits_{x \to \infty} g(x) = \infty$. In fact if g is convex on $(1, \infty)$ and $\lim\limits_{x \to \infty} g(x) = \infty$ then $\frac{g(x)-g(x_1)}{x-x_1} \geq \frac{g(x_2)-g(x_1)}{x_2-x_1}$ for $x_1 < x_2 < x$ and we can choose x_1, x_2 such that $\frac{g(x_2)-g(x_1)}{x_2-x_1} > 0$ so $g(x) \geq ax + b$ for some $a, b \in \mathbb{R}$ with $a > 0$ for x sufficiently large. In particular there is no convex function g on $(1, \infty)$ such that $\lim\limits_{x \to \infty} g(x) = \infty$ and $g(x) \leq \log(1 + x)$ for all x.

Problem 263

Find $\{\int F dF : F$ is a probability distribution on $\mathbb{R}\}$.

The integration by parts formula gives $\int F dF = 1 - \int F(x-) dF(x)$. [This is obtained by evaluating $(F \times F)\{(x, y) : x \leq y\}$ in two ways, using Fubini's Theorem]. Thus $\int F dF \geq 1 - \int F(x) dF(x)$ and $\int F dF \geq \frac{1}{2}$. Note that $\int F dF = \frac{1}{2}$ whenever F is continuous. If dF is the degenerate measure at 0 we get $\int F dF = 1$. The map $F \to \int F dF$ from the space of all complex Borel measures on $\mathbb{R}$ into $\mathbb{R}$ is continuous. In fact $\left|\int F dF - \int G dG\right| \leq \left|\int F dF - \int G dF\right| + \left|\int G dF - \int G dG\right| \leq 2 \|F - G\|$ where $\|F - G\|$ is the total variation norm of $F - G$. The space of probability measures is convex, hence connected. Thus $\{\int F dF : F$ is a probability distribution on $\mathbb{R}\}$ is connected, hence an interval. This proves that the answer is $[\frac{1}{2}, 1]$. An elementary argument to show that any number in $[\frac{1}{2}, 1]$ is of the form $\int F dF$ is as follows: let

$F(x) = 0$ for $x < 0, 1 - [1 - \sqrt{2t-1}]e^{-x}$ for $x \geq 0$ where $t \in [\frac{1}{2}, 1]$ is arbitrary. Then $\int F dF = 2t - 1 + \int\limits_0^\infty \{1 - [1 - \sqrt{2t-1}]e^{-x}\}[1 - \sqrt{2t-1}]e^{-x} dx = t.$

Problem 264

Let $f : (1, \infty) \to (1, \infty)$ be twice differentiable with $xf''(x)$ bounded and $\lim\limits_{x\to\infty} \frac{f(x)}{x} = a$. Show that $\lim\limits_{x\to\infty} f'(x) = a$.

This is a continuous analog of Hardy's Tauberian Theorem. We write $f'(x) - \frac{f(x)}{x}$ as $y\frac{\frac{f(y)}{y} - \frac{f(x)}{x}}{y-x} - \frac{1}{y-x}\int\limits_x^y (y-t)f''(t)dt$ where $y > x$. [This can be proved easily by computing the integral by an integration by parts]. Let $\epsilon > 0$. Choose Δ such that $\left|\frac{f(x)}{x} - a\right| < \epsilon$ if $x > \Delta$. Then $\left|y\frac{\frac{f(y)}{y} - \frac{f(x)}{x}}{y-x}\right| < 2\epsilon\frac{y}{y-x} = 2\epsilon\frac{y/x}{y/x-1}$. Now $\left|\frac{1}{y-x}\int\limits_x^y (y-t)f''(t)dt\right| \leq C\frac{1}{y-x}\int\limits_x^y \frac{y-t}{t}dt$ where $C = \sup\{|xf''(x)| : 1 < x < \infty\}$. Note that $\frac{1}{y-x}\int\limits_x^y \frac{y-t}{t}dt = \frac{1}{y-x}[y\log\frac{y}{x} - (y-x)] = \frac{y/x}{y/x-1}\log\frac{y}{x} - 1 \leq \frac{y/x}{y/x-1}[\frac{y}{x} - 1] - 1 = \frac{y}{x} - 1 \leq \sqrt{\epsilon}$ if $1 < \frac{y}{x} \leq 1 + \sqrt{\epsilon}$. Thus $\left|f'(x) - \frac{f(x)}{x}\right| < 2\epsilon\frac{y/x}{y/x-1} + \sqrt{\epsilon}$ whenever $\Delta < x$ and $x < y \leq x(1+\sqrt{\epsilon})$. Suppose $\frac{y}{x} = 1 + \sqrt{\epsilon}$. Then $x < y \leq x(1+\sqrt{\epsilon})$ and $\left|f'(x) - \frac{f(x)}{x}\right| < 2\epsilon\frac{y/x}{y/x-1} + \sqrt{\epsilon} = 2\epsilon\frac{1+\sqrt{\epsilon}}{\sqrt{\epsilon}} + \sqrt{\epsilon} = \sqrt{\epsilon}[2(1+\sqrt{\epsilon}) + 1]$.

Problem 265

Prove Uniform Boundedness Principle (i.e. Banach Steinhaus Theorem) without using Baire Category Theorem

Let X and Y be Banach spaces and $\{T_i : i \in I\}$ be a family of bounded operators from X into Y such that $\sup\{\|T_i x\| : i \in I\} < \infty$ for each $x \in X$. Suppose, if possible, $\sup\{\|T_i\| : i \in I\} = \infty$. Let $\|T_{i_1}\| > 24$ and $u_1 \in X$ be such that $\|u_1\| = 1$ and $\|T_{i_1}(u_1)\| > \frac{2}{3}\|T_{i_1}\|$. Let $x_1 = \frac{1}{4}u_1$. Then $\|x_1\| = \frac{1}{4}$ and $\|T_{i_1}(x_1)\| > \frac{1}{6}\|T_{i_1}\| = \frac{2}{3}\|T_{i_1}\|\|x_1\|$. Note that $\|T_{i_1}(x_1)\| > 2$. Suppose we have chosen $\{i_1, i_2, ..., i_N\} \subset I$ and $\{x_1, x_2, ..., x_N\} \subset X$ such that $\|x_j\| = \frac{1}{4^j}$, $\left\|T_{i_j}(x_j)\right\| > \frac{2}{3}\left\|T_{i_j}\right\|\|x_j\|$ and $\left\|T_{i_j}(x_j)\right\| > 2\{j + M_{j-1}\}$ where $M_j = \sup\{\|T_i(x_1 + x_2 + ... + x_j)\| : i \in I\}$ for $1 \leq j \leq N$ ($M_0 = 0$). We choose i_{N+1} such that $\left\|T_{i_{N+1}}\right\| > (3)(4^{N+1})(M_N + N + 1)$ and u_{N+1} such that $\left\|T_{i_{N+1}}(u_{N+1})\right\| > \frac{2}{3}\left\|T_{i_{N+1}}\right\|$ and $\|u_{N+1}\| = 1$. Let $x_{N+1} = \frac{1}{4^{N+1}}u_{N+1}$. Then $\|x_{N+1}\| = \frac{1}{4^{N+1}}$, $\left\|T_{i_{N+1}}(x_{N+1})\right\| > \frac{2}{3}\left\|T_{i_{N+1}}\right\|\|x_{N+1}\|$ and $\left\|T_{i_{N+1}}(x_{N+1})\right\| > 2\{N + 1 + M_N\}$. This completes the construction of $\{i_n\}$ and $\{x_n\}$. Let

$x = \sum_{j=1}^{\infty} x_j$. Then $\left\| T_{i_N} \sum_{j=N+1}^{\infty} x_j \right\| \leq \|T_{i_N}\| \sum_{j=N+1}^{\infty} \frac{1}{4^j} = \frac{1}{3} \|T_{i_N}\| \|x_N\|$. Thus

$\|T_{i_N}(x)\| \geq \|T_{i_N}(x_N)\| - \left\| T_{i_N} \sum_{j=N+1}^{\infty} x_j \right\| - \left\| T_{i_N} \sum_{j=1}^{N-1} x_j \right\|$

$\geq \|T_{i_N}(x_N)\| - \frac{1}{3} \|T_{i_N}\| \|x_N\| - M_{N-1} \geq \|T_{i_N}(x_N)\| - \frac{1}{2} \|T_{i_N} x_N\| - M_{N-1} = \frac{1}{2} \|T_{i_N} x_N\| - M_{N-1} \geq n$. This is a contradiction.

Problem 266

Give an example of a sequence of continuous functions $\{f_n\}$ on $[0,1]$ such that $\sup\{\left| \int_a^b f_n(x)dx \right| : n \geq 1.[a,b] \subset [0,1]\} < \infty$ but $\sup\{\int |f_n(x)| dx : n \geq 1\} = \infty$.

We can replace $[0,1]$ by $[-\pi, \pi]$. We take $f_n = D_n$, the $n-th$ Dirichlet kernel. It is well known that $\int |f_n(x)| dx \to \infty$

. Now $\left| \int_a^b f_n(x)dx \right| = \left| \int_a^b \sum_{j=-n}^{n} e^{ijx} dx \right| = \left| \sum_{\substack{j=-n \\ j \neq 0}}^{n} \frac{e^{ijb} - e^{ija}}{ij} + b - a \right| = \left| \sum_{j=1}^{n} \{ \frac{e^{ijb} - e^{ija}}{ij} + \frac{e^{-ijb} - e^{-ija}}{-ij} \} + b - a \right| = \left| \sum_{j=1}^{n} \frac{2i\sin(jb) - 2i\sin(ja)}{ij} + b - a \right|$. From the standard fact (found in most books on Fourier series) $\sum_{j=1}^{n} \frac{\sin(jx)}{j}$ is uniformly bounded.

Problem 267

Let f be a non-negative trigonometric polynomial. Show that thre is a trigonometric polynomial g such that $f = |g|^2$.

Assume first that $f(x) > 0$ for all x. Let $p(z) = z^N \sum_{j=-N}^{N} c_j z^j$ where $f(x) = \sum_{j=-N}^{N} c_j e^{ijx}$. Then g is an entire function. Note that $z^{2N}[p(\frac{1}{\bar{z}})]^- = z^{2N} \frac{1}{z^N} \sum_{j=-N}^{N} \bar{c_j} \frac{1}{z^j} = \sum_{j=-N}^{N} \bar{c_j} z^{N-j} = \sum_{j=-N}^{N} \bar{c}_{-j} z^{N+j} = \sum_{j=-N}^{N} c_j z^{N+j} = p(z)$. [Since f is real valued we have $\sum_{j=-N}^{N} c_j e^{ijx} = \sum_{j=-N}^{N} \bar{c_j} e^{-ijx} = \sum_{j=-N}^{N} \bar{c}_{-j} e^{ijx}$

which implies $\bar{c_{-j}} = c_j$ for all j]. Thus, $p(z) = 0, z \neq 0 \Rightarrow p(\frac{1}{\bar{z}}) = 0$. It follows that $p(z) = c\prod_j (z - a_j)(z - \frac{1}{\bar{a_j}})$ for some none-zero complex numbers $\{a_i\}$ with $c \neq 0$ if we assume, as we may, that $c_N \neq 0$. [The case $c_{-N} \neq 0$] is similar]. Now $e^{-iNx}p(e^{ix}) = f(x)$ and hence $f(x) = e^{-iNx}c\prod_j (e^{ix} - a_j)(e^{ix} - \frac{1}{\bar{a_j}}) = de^{ikx}\prod_j (a_j - e^{ix})(\bar{a_j} - e^{-ix}) = de^{ikx}\prod_j \left|(e^{ix} - a_j)\right|^2$ and k is necessarily 0 because f is non-negative. It follows that $f = |g|^2$ where $g = \sqrt{d}\prod_j (e^{ix} - a_j)$. Now suppose f is allowed to vanish at some points. Then for each $n \geq 1$ there is a trigonometric polynomial g_n such that $f = \left|g_n^2\right|$. The degree of g_n is at most $N/2$, so we may write $g_n = \sum_{j=-N}^{N} c_{j,n}e^{ijx}$. Since $\sum_{j=-N}^{N} \left|c_{j,n}^2\right| = \|g_n\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x) + \frac{1}{n}]dx$ we see that the sequence $\{(c_{-N,n}, c_{-N+1,n}, ..., c_{0,n}, ..., c_{N,n})\}$ has a converfgent subsequence in $\mathbb{C}^{2N+1}$ and so $g_n's$ converge uniformly to a trigonometric polynomial.

Problem 268

Give an example to show that $\frac{\partial}{\partial y}\int_a^b f(x,y)dx$ may not be equal to $\int_a^b \frac{\partial}{\partial y}f(x,y)dx$ even if $\frac{\partial}{\partial y}f(x,y)$ is integrable on $[a,b]$.

Let

$$f(x,y) = \begin{cases} \frac{xy^3}{(x^2+y^2)^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

Then $\frac{\partial}{\partial y}\int_a^b f(x,y)dx = \frac{1}{2}$ at $y = 0$ and $\int_a^b \frac{\partial}{\partial y}f(x,y)dx = 0$ at $y = 0$. In fact $\frac{\partial}{\partial y}f(x,y)$ is identically 0 when $y = 0$!

Problem 269

Prove that there exists a Borel probability measure μ on $\mathbb{R}$ such that $\hat{\mu}$ is differentiable at every point but $\int |x|\, d\mu(x) = \infty$.

It is well known that $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on $\mathbb{R}$ if $a_n \downarrow 0$ and

$na_n \to 0$. [cf. Fourier Series by Edwards]. Let $F_N(x) = \sum_{n=2}^{N} a_n \sin(nx)$ with $a_n = \frac{1}{n \log n}$. Let μ be the discrete proability measure which puts mass $\frac{c}{n^2 \log n}$ at $+n$ and $-n$ for $n = 2, 3, ...,$ c being chosen such that μ is a probability measure. Then $\hat{\mu}(t) = \sum_{n=1}^{\infty} \frac{2c}{n^2 \log n} \cos(nt)$. Let $G_N(t) = \sum_{n=1}^{N} \frac{2c}{n^2 \log n} \cos(nt)$. Then $G_N \to \hat{\mu}(t)$ uniformly and $G_N'(t) = -\sum_{n=1}^{N} \frac{2c}{n \log n} \sin(nt) = -2cF_N(t) \to -2c \sum_{n=1}^{\infty} a_n \sin(nx)$ uniformly. It follows that $\hat{\mu}$ is differentiable at all points and the derivative is $-2c \sum_{n=1}^{\infty} a_n \sin(nx)$.

Problem 270

Find a bounded sequence $\{a_n\}$ which does not converge in Cesaro sense.

Let $n_{k+1} > 3n_k$ and $a_j = (-1)^k \frac{n_{k+1}+n_k}{n_{k+1}-n_k}$ for $n_k \leq j < n_{k+1}$. Then $s_{n_k} = -n_k$ if k is odd and n_k if k is even.

Problem 271

Prove that $\sum_{n=-\infty}^{\infty} e^{-n^2 \pi z} = \frac{1}{\sqrt{z}} \sum_{n=-\infty}^{\infty} e^{n^2 \pi / z}$ for $\operatorname{Re} z > 0$.

Of course, $\sqrt{z}$ is interpreted in the obvious way: it is $e^{\frac{1}{2} Log(z)}$ where $Log(z)$ is the principal branch of logarithm.

This result follows easily by applying the Poisson Summation Formula to the function $e^{-x^2 a/4\pi}$ where $a > 0$ and noting that if the desire formula holds for $z \in (0, \infty)$ it holds for $\operatorname{Re} z > 0$.

Problem 272

Let $H = I_{[0,1/2)} - I_{[1/2,1)}$ and $H_{j,k}(x) = 2^{j/2} H(2^j x - k)$ for $x \in \mathbb{R}, j \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Show that $\{H_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Let $j < j'$. Then $\int H(2^j x - k) H(2^{j'} x - k') dx = \int H(y) H(2^{j'-j} y - 2^{j'-j} k - k') dy$

$= \int_0^{1/2} H(2^{j'-j} y - 2^{j'-j} k - k') dy - \int_{1/2}^{1} H(2^{j'-j} y - 2^{j'-j} k - k') dy$. Let $l = j' - j$ so $l \in \mathbb{N}$. Now $\int_0^{1/2} H(2^{j'-j} y - 2^{j'-j} k - k') dy = \int_0^{1/2} H(2^l y - m) dy$ where $m =$

$2^l k+k'$. But $\int_0^{1/2} H(2^l y-m)dy = \int_{-m}^{2^{l-1}-m} H(z)dz = 0$ because $\int\limits_r^{r+1} H(z)dz = 0$ for any integer r. Similarly, $\int_{1/2}^{1} H(2^{j'-j}y - 2^{j'-j}k - k')dy = \int_{1/2}^{1} H(2^l y - m)dy$ where $m = 2^l k + k'$. But $\int_{1/2}^{1} H(2^l y - m)dy = \int_{2^{l-1}-m}^{2^l-m} H(z)dz = 0$. We have proved that $\{H_{j,k} : j,k \in \mathbb{Z}\}$ is an orthogonal set. Since $H^2 = I_{[0,1)}$ we get $H_{j,k}^2(x) = 2^j I_{[0,1)}(2^j x - k) = 2^j I_{[\frac{k}{2^j}, \frac{k+1}{2^j})}$ and so $\int H_{j,k}^2(x)dx = 1$. Thus $\{H_{j,k} : j,k \in \mathbb{Z}\}$ is orthonormal.

To prove completeness let $f \in L^2(\mathbb{R})$ and $\int f(x)H_{j,k}(x)dx = 0$ for all integers j and k. We have to show that $f = 0$ a.e.. We first show that f is a.e. constant on $[0,1)$. Consider $\int f(x)[I_{[\frac{i-1}{2^n}, \frac{i}{2^n})} - I_{[\frac{i}{2^n}, \frac{i+1}{2^n})}](x)dx = \int f(x)[I_{[0,\frac{1}{2})} - I_{[\frac{1}{2},1)}](2^{n-1}(x - \frac{i-1}{2^n})dx = 2^{-\frac{n-1}{2}} \int f(x)H_{n-1,\frac{i-1}{2}}(x)dx = 0$ provided i is odd. Thus

$$\int f(x)I_{[\frac{i-1}{2^n}, \frac{i}{2^n})}(x)dx = \int f(x)I_{[\frac{i}{2^n}, \frac{i+1}{2^n})}(x)dx \text{ if } i \text{ is odd.}$$

To see that this holds for even values of i also we use induction on n. It is easy to see that the result holds for $n = 1$ and $n = 2$. Now let j be a positive integer and $a = \int f(x)I_{[\frac{2i-2}{2^n}, \frac{2j-1}{2^n})}(x)dx = \int f(x)I_{[\frac{2i-1}{2^n}, \frac{2j}{2^n})}(x)dx$ (by the previous case with $i = 2j-1$). Let $b = \int f(x)I_{[\frac{2i}{2^n}, \frac{2j+1}{2^n})}(x)dx = \int f(x)I_{[\frac{2i+1}{2^n}, \frac{2j+2}{2^n})}(x)dx$ (by the previous case with $i = 2j+1$). We have $2a = \int f(x)I_{[\frac{2i-2}{2^n}, \frac{2j-1}{2^n})}(x)dx + \int f(x)I_{[\frac{2i-1}{2^n}, \frac{2j}{2^n})}(x)dx = \int f(x)I_{[\frac{2i-2}{2^n}, \frac{2j}{2^n})}(x)dx = \int f(x)I_{[\frac{i-1}{2^{n-1}}, \frac{j}{2^{n-1}})}(x)dx$ and $2b = \int f(x)I_{[\frac{2i}{2^n}, \frac{2j+1}{2^n})}(x)dx + \int f(x)I_{[\frac{2i+1}{2^n}, \frac{2j+2}{2^n})}(x)dx = \int f(x)I_{[\frac{2i}{2^n}, \frac{2j+2}{2^n})}(x)dx = \int f(x)I_{[\frac{i}{2^{n-1}}, \frac{j+1}{2^{n-1}})}(x)dx$. However induction hypothesis implies that $\int f(x)I_{[\frac{i-1}{2^{n-1}}, \frac{i}{2^{n-1}})}(x)dx = \int f(x)I_{[\frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}})}(x)dx$ for all i (even or odd) and so $2a = 2b$ and $a = b$. So $\int f(x)I_{[\frac{2i-1}{2^n}, \frac{2j}{2^n})}(x)dx = \int f(x)I_{[\frac{2i}{2^n}, \frac{2j+1}{2^n})}(x)dx$ showing that the desired relation holds for i even.

Now let x and y be Lebsgue points of f in $(0,1)$. Then $f(x) = \lim 2^n \int\limits_{[2^n x]/2^n}^{\{[2^n x]+1\}/2^n} f(t)dt$

and $f(y) = \lim 2^n \int\limits_{[2^n y]/2^n}^{\{[2^n y]+1\}/2^n} f(t)dt$. It follows that $f(x) = f(y)$. Thus f is a constant a.e. on $(0,1)$. Note that $f(x+m)$ is also orthogonal to $H_{j,k}$ for all j and k. We conclude that f is a constant on each of the intervals $(n, n+1)$. Now $0 = \int f(x)H_{-1,k}(x)dx = 2^{-1/2}\int f(x)H(\frac{x}{2}-k)dx$ which gives $\int\limits_{2k}^{2k+1} f(x)dx = \int\limits_{2k+1}^{2k+2} f(x)dx$. Thus, if α_k is the constant value of f on $(k, k+1)$ then $\alpha_{2k} = \alpha_{2k+1}$. This means that f is constant on $(0,2), (2,4), \ldots$ (and $(-2, 0(-4,-2), \ldots)$. A similar argument using the fact that $\int f(x)H_{-2,k}(x)dx = 0$ for all k shows that f is a constant on $(0,4), (4,8), \ldots$ (and similar result on negative real axis). An induction argument now shows that f is a constant on $\mathbb{R}$. Since $f \in L^2$ it follows that $f = 0$ a.e.

Problem 273

Let $H_{j,k}$ be as in Problem 272. For $n = 2^j + k, j = 0, 1, 2, \ldots .k \in \{0, 1, \ldots, 2^j - 1\}$ let $\phi_n(x) = 2^{j/2}H(2^j x - k)$ and define $\phi_0(x) = 1$ for all x. Show that $\{\phi_n\}_{n\geq 0}$ is an orthonormal basis for $L^2[0,1]$.

Define f to be 0 on $\mathbb{R}\backslash[0,1]$ and think of f as a function in $L^2(\mathbb{R})$. If f is orthogonal to each ϕ_n then it is orthogonal to $H_{j,k}$ for $j = 0, 1, 2, \ldots, k = 0, 1, \ldots, 2^j - 1$. If $k < 0$ or $k \geq 2^j$ then $H_{j,k}$ vanishes on $(0,1)$ for any $j \in \mathbb{Z}$. If $j < 0$ and $k \geq 1$ then also $H_{j,k}$ vanishes on $(0,1)$. Thus $\int fH_{j,k} = 0$ in these cases. For $k = 0$ the hypothesis implies $\int f = 0$ and this gives $\int fH_{j,k} = \int\limits_0^1 f - 0 = 0$. It follows from Problem 272 that $f = 0$ a.e.

Problem 274

Show that there is no $f \in L^2(\mathbb{R})$ such that the functions $f_n(x) = f(x-n), n \in \mathbb{Z}$ form an orthonormal basis for $L^2(\mathbb{R})$.

Any $g \in L^2(\mathbb{R})$ is of the type $\sum\limits_{n=-\infty}^{\infty} a_n f(x-n)$ (L^2 sum). This gives $\hat{g}(t) = m(t)\hat{f}(t)$ a.e.. where $m(t) = \sum\limits_{n=-\infty}^{\infty} a_n e^{-int}$. Choosing g such that $\hat{g}$ is continuous and never 0 we conclude that $\{t : \hat{f}(t) \neq 0\}$ is a null set. Now $I_{(-1/2,1/2)}\hat{f}$ belongs to $L^2(\mathbb{R})$ and hence there exists $g \in L^2(\mathbb{R})$ such that $\hat{g} = I_{(-1/2,1/2)}\hat{f}$. But $\hat{g}(t) = m(t)\hat{f}(t)$ a.e.. and this gives $I_{(-1/2,1/2)}\hat{f} = m(t)\hat{f}(t)$ a.e.. Combined with the fact that $\{t : \hat{f}(t) \neq 0\}$ is a null set we get $I_{(-1/2,1/2)} = m(t)$ a.e.. This is a contradiction because m has period 2π.

Problem 275
[This problem extends Problem 274]

Show that there is no $f \in L^2(\mathbb{R})$ such that for some constants α, β with $0 < \alpha < \beta < \infty$ the inequlaities
$\alpha \sum_{n=-\infty}^{\infty} |a_n|^2 \leq \left\| \sum_{n=-\infty}^{\infty} a_n f(x-n) \right\|_2^2 \leq \beta \sum_{n=-\infty}^{\infty} |a_n|^2$ for every finitely non-zero sequence $\{a_n\}$ and such that the closed subspace spanned by the functions $f(x-n), n \in \mathbb{Z}$ is all of $L^2(\mathbb{R})$.

It is a well known fact that the hypothesis implies $\alpha \leq \sum_{n=-\infty}^{\infty} \left| \hat{f}(x+2n\pi) \right|^2 \leq \beta$ a.e. [See p 22 of Wojtaszczyk or p 306 of Pinsky].

Let $g(x) = \sum_{n=-\infty}^{\infty} a_n f(x-n)$ where $\{a_n\} \in l^2$. Then $\hat{g}(t) = m(t)\hat{f}(t)$ where $m(t) = \sum_{n=-\infty}^{\infty} a_n e^{-int}$. We have

$\sum_{n=-\infty}^{\infty} \left| \hat{g}(x+2n\pi) \right|^2 = \sum_{n=-\infty}^{\infty} \left| \hat{f}(x+2n\pi) \right|^2 |m(t)|^2$. We can choose $\{a_n\}$ such that $|m(t)|^2 = \frac{1}{2\pi \sum_{n=-\infty}^{\infty} \left| \hat{f}(x+2n\pi) \right|^2}$ since the right side of this equation is a bounded measurable function (and hence an L^2 function on $[0, 2\pi]$). It then follows that $\sum_{n=-\infty}^{\infty} \left| \hat{g}(x+2n\pi) \right|^2 = \frac{1}{2\pi}$ a.e. which implies (by the converse of the result mentioned above) that $\{g(x-n)\}$ is orthonormal. Also $\hat{f}(t) = [m(t)]^{-1}\hat{g}(t)$ and $[m(t)]^{-1}$ is also a periodic L^2 function. This shows that $f(x-k) = \sum_{n=-\infty}^{\infty} b_n g(x-n-k)$ where $\{b_n\} \in l^2$. It follows that $f(x-k)$ belongs to the closed subspace spanned by $\{g(x-n) : n \in \mathbb{Z}\}$ for each integer k. Combined with the hypothesis this implies that $\{g(x-n) : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ and this is impossible by Problem 274.

Problem 276

Let P be the Borel probability measure on $\mathbb{R}$ with density $\frac{1}{\pi(1+x^2)}$. Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = \frac{1}{2}(x - 1/x)$ if $x \neq 0$ and $T(0) = 0$. Show that T is a measure preserving transformation on $(\mathbb{R}, \mathcal{B}, P)$, i.e. $P \circ T^{-1} = P$.

Let f be a continuous function with compact support. We prove that $\int f dP \circ T^{-1} = \int f dP$. Note that $f(T(x)) = 0$ for $|x|$ sufficiently small. Consider $\int\limits_0^\infty f(\frac{1}{2}(x - 1/x))dP(x) = \frac{1}{2} \int\limits_{-\infty}^{\infty} f(y)dP(y)$. [Note that $\frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi(1+x^2)} \frac{2}{1+\frac{1}{x^2}} dy = \frac{2}{\pi(2+x^2+\frac{1}{x^2})} dy$

$= \frac{2}{\pi(4+4y^2)} dy = \frac{1}{2} dP(y)$ where $y = \frac{1}{2}(x - 1/x)$]. Also $\int\limits_{-\infty}^{0} f(\frac{1}{2}(x - 1/x))dP(x) = \int\limits_0^\infty g(\frac{1}{2}(x - 1/x))dP(x)$ where $g(x) = f(-x)$ and so $\int\limits_{-\infty}^{0} f(\frac{1}{2}(x - 1/x))dP(x) = \frac{1}{2} \int\limits_{-\infty}^{\infty} g(y)dP(y) = \frac{1}{2} \int\limits_{-\infty}^{\infty} f(y)dP(y)$. This completes the proof.

Problem 277

Let $\{a_n\}$ be a bounded sequence of complex numbers and $0 < p < \infty$. Show that $\frac{1}{n} \sum_{k=0}^{n-1} |a_k| \to 0$ if and only if $\frac{1}{n} \sum_{k=0}^{n-1} |a_k|^p \to 0$.

It suffices to show that $\frac{1}{n} \sum_{k=0}^{n-1} |a_k| \to 0$ if and only if there exists $I \subset \{0, 1, 2, ...\}$ such that $\lim\limits_{n \notin I, n \to \infty} a_n = 0$ and $\frac{\#\{I \cap [0,n)\}}{n} \to 0$ as $n \to \infty$. Suppose $\frac{1}{n} \sum_{k=0}^{n-1} |a_k| \to 0$. For $k = 1, 2, ...$ let $I_k = \{n \geq 0 : |a_n| \geq \frac{1}{k}\}$. Claim: $\frac{\#\{I_k \cap [0,n)\}}{n} \to 0$ as $n \to \infty$ for each k. Indeed, this follows from the inequality $\frac{1}{n} \sum_{k=0}^{n-1} |a_k| \geq \frac{\#\{I_k \cap [0,n)\}}{nk}$. There exist integers $0 = n_0 < n_1 < ...$ such that $n \geq n_k$ implies $\frac{\#\{I_k \cap [0,n)\}}{n} < \frac{1}{k}$. Let $I = \bigcup_{k=0}^{\infty} \{I_{k+1} \cap [n_k, n_{k+1})\}$. Let $n_k \leq n < n_{k+1}$. Then $I \cap [0, n) \subset [I_k \cap [0, n_k)] \cup [I_{k+1} \cap [0, n)]$. Hence $\frac{\#\{I \cap [0,n)\}}{n} \leq \frac{\#\{I_k \cap [0,n_k)\}}{n} + \frac{\#\{I_{k+1} \cap [0,n)\}}{n} < \frac{n_k}{n} \frac{1}{k} + \frac{1}{k+1} \leq \frac{1}{k} + \frac{1}{k+1}$. We have proved that $\frac{\#\{I\cap[0,n)\}}{n} \to 0$ as $n \to \infty$. If $n > n_k$ and $n \notin I$ then $n \notin I_{k+1}$ (for, otherwise, there exists $\rho \geq k$ such that $n_\rho \leq n < n_{\rho+1}$ and $n \in I_{k+1} \subset I_{\rho+1}$ so $n \in I_{\rho+1} \cap [n_\rho, n_{\rho+1}) \subset I$ which is a contradiction). Thus $|a_n| < \frac{1}{k+1}$ for $n > n_k$, $n \notin I$ completing the proof of one the 'only if' part. For the 'if' part let $|a_n| \leq C$ and let $\epsilon > 0$. There exisrs n_ϵ such that $|a_n| < \epsilon$ if $n > n_\epsilon$ and $n \notin I$. Also there exists m_ϵ such that $\frac{\#\{I\cap[0,n)\}}{n} < \epsilon$ if $n > m_\epsilon$. For $n > \max\{n_\epsilon, m_\epsilon\}$ we have $\frac{1}{n}\sum_{k=0}^{n-1} |a_k| < \epsilon + \epsilon C$.

[We have proved a stronger result than what was stated. In particular we have proved that $\frac{1}{n}\sum_{k=0}^{n-1} |a_k| \to 0$ implies $\frac{1}{n}\sum_{k=0}^{n-1} f(a_k) \to 0$ for any bounded positive function function f continuous at 0. See Problem 294 for a related result].

Problem 278

Show that any positive linear operator T from $L^p(\mu)$ (where $1 \leq p < \infty$) into itself is bounded.

If T is not bounded the there exists $\{f_n\} \subset L^p(\mu)$ such that $\|f_n\|_p = 1$ and $\|Tf_n\|_p > n^2$. We may also assume that $f_n's$ are non-negative. Let $f = \sum_{k=1}^{\infty} \frac{f_n}{n^2}$. [The series converges in L^p]. We have $\int (Tf)^p d\mu \geq \int (T\sum_{k=1}^{N} \frac{f_n}{n^2})^p d\mu \geq \sum_{k=1}^{N} \int (\frac{Tf_n}{n^2})^p d\mu > \sum_{k=1}^{N} \frac{n^{2p}}{n^{2p}} = N$ for every N which is a contradiction. [We have used the fact that $(\sum_{k=1}^{N} a_n)^p \geq \sum_{k=1}^{N} a_n^p$ for all non-negative numbers a_n which follows from the fact that $t \to t^i + (1-t)^p, 0 \leq t \leq 1$ attains its maximum at $t = \frac{1}{2}$ and the maximum value is ≤ 1, so $a^p + b^p \leq (a+b)^p$ for $a, b \geq 0$].

Problem 279

[This is a standard result that follows from approximation of irrationals by rationals]

If α is a an irrational number in $(0,1)$ show that the numbers $\{n\alpha (\mathrm{mod}\, 1) : n \in \mathbb{N}\}$ is dense in $(0,1)$. [Equivalently, if c is a complex number which is not a root of unity such that $|c| = 1$ then the set $\{1, c, c^2, ...\}$ is dense in the unit circle].

By a well known result in Number Theory [cf. Hardy and Wright: Theory of Numbers] we can find positive integers p_n, q_n such that $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$. Note that $p_n, q_n \to \infty$ and $\alpha - \frac{p_n - [\sqrt{q_n}]}{q_n} = \frac{[\sqrt{q_n}]}{q_n} - (\alpha - \frac{p_n}{q_n}) > \frac{[\sqrt{q_n}]}{q_n} - \frac{1}{q_n^2} > 0$

and $\alpha - \frac{p_n - [\sqrt{q_n}]}{q_n} < \frac{[\sqrt{q_n}]}{q_n} + \frac{1}{q_n^2} \to 0$. This shows we can assume (by changing $p_n's$ if necessary that) $\frac{p_n}{q_n} \uparrow \alpha$. Now $0 < q_n\alpha - p_n = q_n[\alpha - \frac{p_n}{q_n}] < \frac{1}{q_n}$. It follows that the set $\{n\alpha (\mathrm{mod}\, 1) : n \in \mathbb{N}\}$ contains a sequence $\{r_n\}$ decreasing (strictly) to 0. Now let $0 < a < b < 1$. Let $r \in \{n\alpha (\mathrm{mod}\, 1) : n \in \mathbb{N}\}$ with $0 < r < b - a$. The interval $(\frac{a}{r}, \frac{b}{r})$ has length $\frac{b-a}{r} > 1$ and hence it contains an integer k. Since $\frac{a}{r} > 0$ it is clear that k is a positive integer. Thus kr is a point of (a, b) which belongs to $\{n\alpha (\mathrm{mod}\, 1) : n \in \mathbb{N}\}$.

Second proof: let $|c| = 1$ and assume that $c^n \neq 1$ for any positive integer n. Since $c, c^2, c^3, ...$ are all distinct and this sequence has a convergent subsequence. This subsequenec is Cauchy, so given $\epsilon \in (0, 1)$, we can find $1 \leq n < m$ such that $|c^n - c^m| < \epsilon$. Let $c = e^{it}, 0 \leq t < 2\pi$ and consider the points $e^{itn}, e^{itn+i(m-n)}, e^{itn+2i(m-n)}, ..., e^{itn+iN(m-n)}$. Of course there is a smallest integer N such that $tn + N(m - n) \geq 2\pi$. The distance between any two points consecutive points of $\{e^{itn}, e^{itn+i(m-n)}, e^{itn+2i(m-n)}, ..., e^{itn+iN(m-n)}\}$ is less than ϵ. This gives a finite subset of $\{c, c^2, c^3, ...\}$ such that any point of S^1 is at distance less than ϵ from this finite set.

[More precisely, if $0 \leq s < 2\pi$ then $tn + l(m-n) \leq s < tn + (l+1)(m-n)$ for some l and $\left|e^{is} - e^{tn+l(m-n)}\right| \leq \left|e^{tn+l(m-n)} - e^{tn+(l+1)(m-n)}\right| < \epsilon$ since $0 \leq \alpha < \beta < \gamma < 2\pi, \left|e^{i\alpha} - e^{i\gamma}\right| < \epsilon \Rightarrow \left|e^{i\alpha} - e^{i\beta}\right|^2 = 2 - 2\cos(\alpha - \beta) \leq 2 - 2\cos(\alpha - \gamma) = \left|e^{i\alpha} - e^{i\gamma}\right|^2$ where we used that fact that $2 - 2\cos(\alpha - \gamma) < \epsilon^2$ so $\cos(\alpha - \gamma) > 1 - \epsilon^2/2 > 0$ which implies $|\alpha - \gamma| < \pi/2$ and hence that $\cos(\alpha - \beta) \geq \cos(\alpha - \gamma)$]

Problem 280

Using the fact that any continuous additive map from $\mathbb{R}$ into itself is constant times the identity give an elemenatry proof of the fact that the circle group has only two continuous automorphisms, the maps $a \to \frac{1}{a}$ and the identity. Also find all continuous homomorphisms of S^1.

Remark: Problems 612 and 613 have stronger results with different proofs.

Let $T : S^1 \to S^1$ be a continuous homomorphism. Note that $T(1) = 1$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = T(e^{2\pi it})$. Then f is continuous, $f(0) = 1$ and never vanishes. Fix a positive integer N. On $[-N, N]$ we can find a continuous map g_N (a continuous 'logarithm' of f)such that $f(t) = e^{2\pi i g_N(t)}, -N \leq t \leq N$ and $g_N(0) = 0$. [Logarithms exist locally on S^1 and we can patch up local logarithms using compactness of $[-N, N]$]. Now g_N are consistently defined in the sense $g_N = g_{N+1}$ on $[-N, N]$. [$e^{2\pi i g_N(t)} = e^{2\pi i g_{N+1}(t)}$ shows that $g_{N+!} - g_N$ is an integer valued continuous function, hence a constant. Since it vanishes at 0 we get $g_N = g_{N+1}$ on $[-N, N]$]. We conclude that there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $T(e^{2\pi it}) = e^{2\pi i g(t)}$ and $g(0) = 0$. Since g is continuous and $g(t + s) - g(t) - g(s)$ is integer valued [because $e^{2\pi i[g(t+s)-g(t)-g(s)]} = 1$] we see that g is a continuous additive real function on $\mathbb{R}$ and hence there is a real number a such that $g(t) = at$ for all t. Thus $T(e^{2\pi it}) = f(t) = e^{2\pi ig(t)} = e^{2\pi iat}$. Now note that $1 = T(1) = T\{(e^{\pi i})^2\} = \{T(e^{\pi i})\}^2 = e^{2\pi ia}$. Hence a is an integer, say m and $T(z) = z^m \; \forall z \in S^1$. In particular, if T is a bijection then $m = \pm 1$ and $T(z) \equiv z$ or $T(z) \equiv z^{-1}$.

Problem 281

Find all continuous automorphisms of the torus $S^1 \times S^1$ (coordinate-wise multiplication)

Let T be an automorphism of the torus $S^1 \times S^1$(which is a group under coordinatewise multiplication). Let $T_1(z)$ be the first coordinate of $T(z,1)$ and $T_2(z)$ be the second coordinate of $T(z,1)$. Let $T_3(z)$ be the first coordinate of $T(1,z)$ and $T_4(z)$ be the second coordinate of $T(1,z)$. Then T_j is a homomorphism of S^1 for $j = 1,2,3,4$. Hence there exist integers j,k,n,m such that $T_1(z) = z^j, v, T_2(z) = z^k, T_3(z) = z^n, T_4(z) = z^m$. It follows that $T(a,b) = T(a,1)T(1,b) = (a^j, a^k)(b^n, b^m) = (a^j b^n, a^k b^m)$. We have to determine when this map is an automorphism. If T is an automorphism the so is T^{-1} and so $T^{-1}(a,b) = (a^{j'} b^{n'}, a^{k'} b^{m'})$ for some integers j', n', k', m'. We now have $(a,b) = TT^{-1}(a,b) = (a^{jj'+k'n} b^{jn'+m'n}, a^{j'k+k'm} b^{n'k+m'm}) \forall a,b \in S^1$. This implies $jj' + k'n = 1, jn' + m'n = 0, j'k + k'm = 0$ and $n'k + m'm = 1$. In other words $\begin{pmatrix} n & j \\ m & k \end{pmatrix} \begin{pmatrix} k' & m' \\ j' & n' \end{pmatrix} = 1$. Taking determinants and noting that the determinants of the two matrices are integers we conclude that $nk - mj = \pm 1$. Conversely suppose $nk - mj = \pm 1$. The inverse of $\begin{pmatrix} n & j \\ m & k \end{pmatrix}$ has integer entries because the detrminant is ± 1 and the adjoint has integer entries. Thus there is a transformation of the type $S(a,b) = (a^{j'} b^{n'}, a^{k'} b^{m'})$ with $TS = I = ST$. It follows that T is bijective with an inverse which is also a homomorphism. The inverse is automatically continuous.

Problem 282

Let X be a compact Hausdorff space. Show that $C(X)$ is separable if and only if X is metrizable.

Suppose $C(X)$ has a countable dense subset $\{f_n\}$. Define $d(x,y) = \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n [1 + |f_n(x) - f_n(y)|]}$. Since $\{f_n\}$ separates points it is clear that d is a metric on X. Consider the identity map : $X \to X$ where the domain is given the original topology and the range is given the metric topology. To prove that this map is continuous consider an open ball $B(x,r)$ in (X,d). Let y be in this ball. Choose N such that $\frac{1}{2^N} < r - d(x,y)$. We claim that if $z \in \bigcap_{i=1}^{N} f_i^{-1}\{t : |t - f_i(y)| < \epsilon\}$ then $z \in B(x,r)$ provided ϵ is sufficiently small. Since $\bigcap_{i=1}^{N} f_i^{-1}\{t : |t - f_i(y)| < \epsilon\}$

is an open set in the given topology which contains y we can conclude that the identity map is continuous; its inverse is automatically continuous since X is Hausdorff with the metric topology and compact with the original topology. Thus we can conclude that d metrizes the given topology on X completing one part of the statement. So let $z \in \bigcap_{i=1}^{N} f_i^{-1}\{t : |t - f_i(y)| < \epsilon\}$. Then

$$d(x,z) \leq d(x,y) + \sum_{n=1}^{\infty} \frac{|f_n(z)-f_n(y)|}{2^n[1+|f_n(z)-f_n(y)|]} < d(x,y) + \frac{1}{2^N} + \sum_{n=1}^{N} \frac{|f_n(z)-f_n(y)|}{2^n[1+|f_n(z)-f_n(y)|]}$$

$< d(x,y) + \frac{1}{2^N} + \frac{\epsilon}{1+\epsilon} \sum_{n=1}^{N} \frac{1}{2^N} < r$ if $\frac{\epsilon}{1+\epsilon} < r - d(x,y) - \frac{1}{2^N}$. We now prove the other half of the statement. Let d be a metric for the topology of X. Let $\{x_n\}$ be a countable dense subset of X. For each n open balls of radius $\frac{1}{n}$ around the points x_i cover X and hence there exists an integer k_n such that $X = \bigcup_{i=1}^{k_n} B(x_i, \frac{1}{n})$. For each n and $i \leq k_n$ there is a continuous function $f_{i,n} : X \to [0,1]$ such that $f_{i,n} = 1$ on the closed ball around x_i with radius $\frac{1}{2n}$ and 0 in the complement of the ball $B(x_i, \frac{1}{n})$. Let M be the collection of all finite linear combinations of finite products of the functions $\{f_{i,n}\}$. This is an algebra. If we show that the functions $f_{i,n.}$ separate points of X it would follow by Stone-Weierstrass Theorem that M is dense in $C(X)$ and we can then conclude that rational linear combinations of finite products of the functions $\{f_{i,n}\}$ give us a countable dense subset of $C(X)$. If $x \neq y$ then there exists n such that $\frac{2}{n} < d(x,y)$. If $x \in B(x_i, \frac{1}{n})$ then $f_{in}(x) = 1$ and $f_{i,n}(y) = 0$. This completes the proof.

Problem 283

Prove or disprove the following:

1) if f is monotonically increasing on $[a,b]$ then we can find continuous functions f_n, g_n $(n = 1,2,...)$ such that $\phi_n \leq f \leq g_n, g_n \downarrow f$ and $\phi_n \uparrow f$ pointwise.

2) if f is Riemann integrable on $[a,b]$ then we can find continuous functions ϕ_n, g_n $(n = 1,2,...)$ which are uniformly bounded such that $\phi_n \leq f \leq g_n$ and $g_n - \phi_n \to 0$ almost everwhere.

The first statement is false: $f = \inf\{g_n : n \in \mathbb{N}\}$ implies that f is upper semi-continuous and $f = \sup\{f_n : n \in \mathbb{N}\}$ implies that f is is lower semi-continuous. Thus f is necessarily continuous

2) is true. Let $\{t_i : 1 \leq i \leq k\}$ be a partition of $[a,b]$. Let $g = \sum_{i=1}^{k} m_i I_{[t_{i-1},t_i)}$ and $h = \sum_{i=1}^{k} M_i I_{[t_{i-1},t_i)}$ where m_i and M_i are the infimum and the supremum

of f on $[t_{i-1}, t_i]$. Let $\epsilon > 0$. Let $1 \leq i \leq k$. If $M_{i-1} \leq M_i$ we modify h on a smal interval to the left of t_i and if $M_{i-1} > M_i$ we modify it on the right in such a way that the modified function still dominates f and it is continuous at t. By this procedure we can find piece-wise linear continuous functions G and H such that $G \leq f \leq H$ and $m\{x : g(x) \neq G(x)\} < \epsilon, m\{x : h(x) \neq H(x)\} < \epsilon$. Taking a sequence of partitions such that $h_n - g_n \to 0$ a.e. for the corresponding functions g_n and h_n and then choosing piece-wise linear continuous functions G_n, H_n with $m\{x : G_n \neq g_n\} < \frac{1}{2^n}$ and $m\{x : H_n \neq h_n\} < \frac{1}{2^n}$. If $x \notin \limsup\{x : G_n \neq g_n\} \cup \lim m \sup\{x : H_n \neq h_n\}$ and $g_n(x) - h_n(x) \to 0$ then we get $G_n(x) - H_n(x) \to 0$. It is known that $g_n(x) - h_n(x) \to 0$ a.e. and this completes the proof. [A trivial modification makes the functions G_n, H_n uniformly bounded].

Problem 284

Use previous problem to prove the following result of Weyl:

if $f \in C[0,1]$ and $T(x) = x + \alpha \bmod(1)$ where α is irrational the $\frac{1}{n}\sum\limits_{k=0}^{n-1} f(T^k x) \to \int\limits_0^1 f(x)dx$ uniformly for any Riemann integrable function f.

For continuous f this result is proved using The Ergodic Theorem and a Functional Analytic argument (which is outlined below) For f Riemann integrable choose approximating continuous functions as in part 2) of previuos problem. We have $\frac{1}{n}\sum\limits_{k=0}^{n-1} \phi_j(T^k x) \to \int\limits_0^1 \phi_j(x)dx$ and $\frac{1}{n}\sum\limits_{k=0}^{n-1} g_j(T^k x) \to \int\limits_0^1 g_j(x)dx$ uniformly for each fixed j. Note that $\int\limits_0^1 \phi_j(x)dx$ and $\int\limits_0^1 g_j(x)dx$ both converge to $\int\limits_0^1 f(x)dx$ by Bounded Convergence Theorem. These facts, together with the inequalities $\frac{1}{n}\sum\limits_{k=0}^{n-1} \phi_j(T^k x) \leq \frac{1}{n}\sum\limits_{k=0}^{n-1} f(T^k x) \leq \frac{1}{n}\sum\limits_{k=0}^{n-1} g_j(T^k x)$ clearly imply $\frac{1}{n}\sum\limits_{k=0}^{n-1} f(T^k x) \to \int\limits_0^1 f(x)dx$ uniformly.

[We now sketch a proof of the result for continuous f. Suppose the result is false. Then there exists $\epsilon > 0, n_j \uparrow \infty$ and points $x(j)$ such that $\left|\frac{1}{n_j}\sum\limits_{k=0}^{n_j-1} f(T^k x(j)) - \int\limits_0^1 f\right| > \epsilon$ for all j. Let $P_j = \frac{1}{n_j}\sum\limits_{k=0}^{n_j-1} \delta_{T^k(x(j))}$. By separability of $C[0,1]$ and Banach-Alaoglu Theorem we can find a subsequence $\{j_l\}$ and a measure Q such that $\int hdP_{j_l} \to \int hdQ$ for any $h \in C[0,1]$. Since $\int hdQ \geq 0$ for non-negative h and since $\int 1dQ = \lim \int 1dP_{j_l} = 1$ we see that Q is a probability measure. By definition of the measures P_j this gives $\frac{1}{n_j} \sum_{k=0}^{n_j-1} h(T^k x(j)) \to \int_0^1 hdQ$. Hence $\int_0^1 hdQ \circ T^{-1} = \int_0^1 h(T(y))dQ(y) = \lim_{j\to\infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} h(T \circ T^k x(j)) = \lim_{j\to\infty} \frac{1}{n_j} \sum_{k=1}^{n_j} h(T^k x(j)) = \lim_{j\to\infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} h(T^k x(j))$ (since h is bounded) and this gives $\int_0^1 hdQ \circ T^{-1} = \int_0^1 hdQ$ for any continuous function h. Taking $h(x) = e^{2\pi inx}$ we get $\int_0^1 e^{2\pi in(x+\alpha)} dQ(x) = \int_0^1 e^{2\pi inx} dQ(x)$ by the definition of T and we get $e^{2\pi in\alpha} \int_0^1 e^{2\pi inx} dQ(x) = \int_0^1 e^{2\pi inx} dQ(x)$. Since α is irrational this gives $\int_0^1 e^{2\pi inx} dQ(x) = 0$ for all $n \neq 0$. Bu then $\int_0^1 e^{2\pi inx} dQ(x) = \int_0^1 e^{2\pi inx} dx$ for all $n \neq 0$ and hence $\int_0^1 pdQ = \int_0^1 p(x)dx$ for any trigonometric polynomial p. It follows from Fejer's Theorem that the same holds for all continous functions p. Thus $Q(x)$ is nothing but the Lebesgue measure on $[0,1]$. However $\left| \frac{1}{n_j} \sum_{k=0}^{n_j-1} f(T^k x(j)) - \int_0^1 f \right| > \epsilon$ for all j implies that $\left| \int fdQ - \int_0^1 f \right| \geq \epsilon$ which is a contradiction.

PROBLEM 285 [BY K B ATHREYA]

Let $\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{\log k - [\log k]}$ where log denotes logarithm to base 2 and $[x]$ is the greatest integer not exceeding x. Does this sequence of probability measures on $[0,1]$ converge weakly?

[Weak convegence here is actually weak* convergence in $C^*[0,1]$]

We claim that $\mu_n \to \mu$ where $d\mu(x) = (\log_e 2)2^x dx$. Bt Stone-Weiertstrass Theorem it suffices to show that $\int fd_n \to \int fd\mu$ if $f(x) = 2^{cx}$ where c is a real number. For this f we have $\int f d_n = \frac{1}{n}\sum_{k=1}^{n} 2^{c[\log k - [\log k]]} = \frac{1}{n}\sum_{m=0}^{\log n - 1}\sum_{k=2^m}^{2^{m+1}-1} k^c 2^{-mc} +$ $\frac{1}{n}2^{c[\log n - [\log n]]}$. Of course, the last term tends to 0. Now, assuming that $c \geq 0$, $\sum_{k=2^m}^{2^{m+1}-1} k^c \leq \int_{2^m}^{2^{m+1}} x^c dx = \frac{2^{(m+1)(c+1)} - 2^{m(c+1)}}{c+1} = 2^{mc}2^m \frac{2^{c+1}-1}{c+1}$ and so $\frac{1}{n}\sum_{m=0}^{\log n - 1}\sum_{k=2^m}^{2^{m+1}-1} k^c 2^{-mc} \leq \frac{1}{n}\sum_{m=0}^{\log n - 1} 2^m \frac{2^{c+1}-1}{c+1}$

$= \frac{1}{n}(n-1)\frac{2^{c+1}-1}{c+1} \to \frac{2^{c+1}-1}{c+1}$ as $n \to \infty$. Further $\sum_{k=2^m}^{2^{m+1}-1} k^c \geq \int_{2^m-1}^{2^{m+1}-1} x^c dx =$ $\frac{\{2^{m+1}-1\}^{c+1} - \{2^m - 1\}^{c+1}}{c+1}$

$= 2^{mc}2^m \frac{1}{c+1}\{(2-\frac{1}{2^m})^{c+1} - (1-\frac{1}{2^m})^{c+1}\}$ and $\{(2-\frac{1}{2^m})^{c+1} - (1-\frac{1}{2^m})^{c+1}\} \to$ $2^{c+1} - 1$ which shows that $\liminf \frac{1}{n}\sum_{m=0}^{\log n - 1}\sum_{k=2^m}^{2^{m+1}-1} k^c 2^{-mc} \geq \frac{2^{c+1}-1}{c+1}$. We have proved that $\int 2^{cx} d\mu_n(x) \to \frac{2^{c+1}-1}{c+1} = \int 2^{cx} d\mu(x)$ for all $c \geq 0$. A similar argument holds for $c < 0$.

Problem 286

Let P be a Borel probability measure on a compact metric space X such that $P(A) = 0$ or 1 for any Borel set A. Show that $p = \delta_x$ for some $x \in X$.

For each n let $\{A_{n.i}\}$ be a partition of X into sets of diameter at most $\frac{1}{n}$. By hypothesis there exists i_n such that $P(A_{i_n}) = 1$. Let C_n be the closure of A_{i_n}. Then $P(C_n) = 1$ and hence $P(C_1 \cap C_2 \cap ... \cap C_n) = 1$ too. The family $\{C_n\}$ therefore has finite intersection property and hence there is a point x in their intersection. But the diameter of C_n tends to 0 so $\{x\} = \cap C_n$. It follows that $P\{x\} = 1$.

Problem 287

Let f and g be non-negative measurable functions on $[0,1]$ such that $\int_E f(x)dx < \infty \Rightarrow \int_E g(x)dx < \infty$. Show that $g \leq Cf + h$ for some non-negative integrable function h and some $C \in (0,\infty)$.

We prove the following stronger result: if μ and ν are positive non-atomic mesures on $(\Omega, \mathcal{F})$ such that $\mu(E) < \infty \Rightarrow \nu(E) < \infty$. Then there exists a finite positive measure λ and $C \in (0,\infty)$. such that $\nu(E) \leq C\mu(E) + \lambda(E)$ for all $E \in \mathcal{F}$. Once this result is proved we can take $\mu(E) = \int\limits_E f, \nu(E) = \int\limits_E g$ to get $\int\limits_E g \leq C \int\limits_E f + \lambda(E)$ and define $h = \max\{g - Cf, 0\}$ so that $g \leq Cf + h$. Note that $\int\limits_E (g - Cf) \leq \lambda(E)$ for all E which implies $\int\limits_E (g - Cf)^+ \leq \lambda(E \cap \{g - Cf > 0\}) \leq \lambda(E)$. Thus $\int h \leq \lambda([0,1]) < \infty$. Now suppose μ and ν are as above. We claim that there exists $C \in (0, \infty)$ such that $\mu(E) \leq 2$ implies $\nu(E) \leq C$. If this is false then we can find sets $\{E_n\}$ such that $\mu(E_n) \leq 2$ but $\nu(E_n) \geq 2^{n+1} + \sum\limits_{k=1}^{n-1} \nu(E_k)$ and $\nu(E_n) < \infty$. [The last sum is taken to be 0 when $n = 1$]. Let $A_n = E_n \backslash \{E_1 \cup E_2 \cup ... \cup E_{n-1}\}$ so that $A_n's$ are disjoint, $\mu(A_n) \leq 2$ and $\infty > \nu(A_n) \geq \nu(E_n) - \sum\limits_{k=1}^{n-1} \nu(E_k) \geq 2^{n+1}$. We can write A_n as a disjoint union $\bigcup\limits_{j=1}^{2^n} A_{n,j}$ with $\nu(A_{n,j}) = 1$ for $1 \leq j < 2^n$ and $\nu(A_{n,2^n}) \geq 2^{n+1} - (2^n - 1) \geq 1$. Since $\sum\limits_{j=1}^{2^n} \mu(A_{n,j}) = \mu(A_n) \leq 2$ we can find $j_n \leq 2^n$ with $\mu(A_{n,j_n}) \leq \frac{2}{2^n} = 2^{1-n}$. Let $A = \bigcup\limits_{j=1}^{\infty} A_{n,j_n}$. Then $\mu(A) \leq \sum\limits_{j=1}^{\infty} 2^{1-n} = 2$ but $\nu(A) = \sum\limits_{j=1}^{\infty} \nu(A_{n,j_n}) \geq \sum\limits_{j=1}^{\infty} 1 = \infty$ contradicting the hypothesis. This proves our claim. We now show that $\nu(E) \leq C\mu(E) + C$ for all E. We may suppose $\mu(E) < \infty$. Let $n - 1 \leq \mu(E) < n$. Write E as a disjoint union $\bigcup\limits_{j=1}^{n} E_j$ with $\mu(E_j) = 1$ for $j < n$ and $\mu(E_m) \leq 1$. We have $\nu(E) \leq \sum\limits_{j=1}^{n} \nu(E_j) \leq Cn$ (by the claim). Hence $\nu(E) \leq Cn = C \sum\limits_{j=1}^{n-1} \mu(E_j) + C \leq C(\mu(E) + 1)$ as asserted. Let $\lambda = (\nu - C\mu)^+$. λ is a positive measure and $\lambda(E) \geq \nu(E) - C\mu(E)$ so $\nu \leq C\mu + \lambda$. Also $\lambda(E) = (\nu - C\mu)(E \cap F)$ where $\{F, F^c\}$ is a Hahn decomposition of the signed measure $\nu - C\mu$. So $\lambda(E) \leq C$ for ane E proving that λ is a finite measure.

Problem 288

Let X be a non-empty set and let τ be the smallest topology that makes a given function $f : X \to \mathbb{R}$ continuous. Let $g : (X, \tau) \to \mathbb{R}$ be continuous. Prove or disprove that there exists a continuous function $h : \mathbb{R} \to \mathbb{R}$ with $g = h \circ f$. Answer the same question for measurable functions.

For measurable functions the result is true and the proof follows easily from simple function approximation. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = e^{-|x|}$ and $g(x) = e^{|x|}$ then $g = \frac{1}{f}$ and hence g is continuous w.r.t. the topology generated by f. However if $h : \mathbb{R} \to \mathbb{R}$ is continuous and $g = h \circ f$ then $h(t) = \frac{1}{t}$ for all $t \in (0, 1]$ which contradicts continuity of h at 0. Note however that under the hypothesis of the problem $f(x) = f(y) \Rightarrow g(x) = g(y)$ and we can define h on the range of f by $g = h \circ f$. Further it is easy to see that h is continuous on the range of f. Extending h to a continuous function on $\mathbb{R}$ is not always possible.

Problem 289

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that the TFE:
1) $L^p \subset L^q$ for some $p < q$ in $(0, \infty)$
2) $\inf\{\mu(E) : \mu(E) > 0\} > 0$
3) $L^p \subset L^q$ for all $p < q$ in $(0, \infty)$

We prove 1) implies 2) and 2) implies 3). Since 3) obviuously implies 1) the proof would be complete. Let 1) hold. Note that $L^{p\alpha} \subset L^{q\alpha}$ for all $\alpha \in (0, \infty)$ so we may suppose $p \geq 1$. Let $T : L^p \to L^q$ be the inclusion map. This linear map has closed graph. Hence it is bounded. Let $\|f\|_q \leq C \|f\|_p$ for all $f \in L^p$ hence for all measurable f. Then $\mu^{1/q}(E) \leq C\mu^{1/p}(E)$ for all E. We get $\inf\{\mu(E) : \mu(E) > 0\} \geq C^{-\{\frac{1}{p}-\frac{1}{q}\}^{-1}}$ and 2) holds. Now let 2) hold and $0 < p < q < \infty$. Let $f \in L^p$. Let $A_n = \{|f| > n\}$. Since $\mu(A_n) \leq n^{-p} \int |f|^p \, d\mu \to 0$ we conclude from 2) that $\mu(A_n) = 0$ for n sufficiently large. Thus $f \in L^\infty$ and $\int |f|^q \, d\mu \leq \|f\|_\infty^{q-p} \int |f|^p \, d\mu < \infty$

Problem 290

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that the TFE:
1) $L^q \subset L^p$ for some $p < q$ in $(0, \infty)$
2) $\sup\{\mu(E) : \mu(E) < \infty\} < \infty$
3) $L^q \subset L^p$ for all $p < q$ in $(0, \infty)$

Let 1) hold. As in Problem 289 we may suppose $p \geq 1$ and use Closed Graph Theorem to conclude that $\|f\|_p \leq C \|f\|_q$ for all $f \in L^q$ hence for all measurable f. Thus $\mu^{1/p}(E) \leq C\mu^{1/q}(E)$ for all E. It follows that $\mu(E) \leq C^{\{\frac{1}{p}-\frac{1}{q}\}^{-1}}$ whenever $\mu(E) < \infty$ so 2) holds. Thus 1) implies 2). Now let 2) hold. Let $f \in L^q$ and $A_n = \{\frac{1}{n+1} \leq |f| < \frac{1}{n}\}$. Then $\mu(A_n) < \infty$ and hence $\mu(A_n) \leq C \equiv \sup\{\mu(E) : \mu(E) < \infty\}$. Also $\mu(\bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu(A_j) < \infty$ and

2) implies $\sum_{j=1}^{n}\mu(A_j) = \mu(\bigcup_{j=1}^{n} A_j) \leq C$ for all n. Thus $\sum_{j=1}^{\infty}\mu(A_j) < \infty$. Now $\int |f|^p \, d\mu = \int_{\{|f|\geq 1\}} |f|^p \, d\mu + \sum_{j=1}^{\infty}\int_{A_j} |f|^p \, d\mu$

$\leq \int_{\{|f|\geq 1\}} |f|^q \, d\mu + \sum_{j=1}^{\infty} j^{-p}\mu(A_j) \leq \int |f|^q \, d\mu + \sum_{j=1}^{\infty}\mu(A_j) < \infty$. Thus 2) implies 3). Of course 3) implies 1).

Problem 291

For a measure space $(\Omega, \mathcal{F}, \mu)$ TFE:
1) $L^p \nsubseteq L^q$ whenever p and q are distinct numbers in $(0, \infty)$
2) $\inf\{\mu(E) : 0 < \mu(E) < \infty\} = 0$ and $\sup\{\mu(E) : 0 < \mu(E) < \infty\} = \infty$
3) for any convex set $I \subset (0, \infty)$ there exists a measurable function f on Ω such that $\{p \in (0, \infty) : \int |f|^p \, d\mu < \infty\} = I$

Let 2) hold. We shall construct measurable non-negative functions $f_i, i = 1, 2, 3, 4$ such that $\{p \in (0, \infty) : \int |f_1|^p \, d\mu < \infty\} == (0, 1), \{p \in (0, \infty) : \int |f_2|^p \, d\mu < \infty\} = (0, 1]$,

$\{p \in (0, \infty) : \int |f_3|^p \, d\mu < \infty\} = (1, \infty)$ and $\{p \in (0, \infty) : \int |f_4|^p \, d\mu < \infty\} = [1, \infty)$. Once such functions are constructed we can easily conclude that 3) holds: just note that the collection $\mathcal{I}$ of all convex sets $I \subset (0, \infty)$ for which there exists a measurable function f on Ω such that $\{p \in (0, \infty) : \int |f|^p \, d\mu < \infty\} = I\}$ has the following properties: $I_1.I_2 \in \mathcal{I}$ implies $I_1 \cap I_2 \in \mathcal{I}$ and $I \in \mathcal{I}$ implies $\alpha I \in \mathcal{I}$ for any $\alpha \in (0, \infty)$. We first note that there exists sets $E_n, n = 1, 2, ...$such that $0 < \mu(E_{n+1}) < \frac{1}{2}\mu(E_n) < \infty$. Thus $\mu(E_{n+k}) < (\frac{1}{2})^k \mu(E_n)$ and $\sum \mu(E_n) < \infty$. If $F_n = E_n \backslash \{E_{n+1} \cup E_{n+2} \cup ...\}$ then $F_n' s$ are disjoint and $\mu(F_n) \geq \mu(E_n) - \mu(E_n \cap \{E_{n+1} \cup E_{n+2} \cup ...\}) \geq \mu(E_n) - \sum_{j=n+1}^{\infty} \mu(E_j) > \mu(E_n)(1 - \sum_{j=n+1}^{\infty} (\frac{1}{2})^{j-n}) = 0$. Thus $F_n' s$ are disjoint sets of positive measure. Let $x_n = \sum_{j=n}^{\infty} \mu(F_j)$. Let $f_1 = \sum x_n^{-1} I_{F_n}$ and $f_2 = \sum \frac{1}{x_n [1 + \log^2 x_n]} I_{F_n}$. If $0 < p < 1$ then $\int f_1^p d\mu = \sum x_n^p \mu(F_n) = \sum x_n^{-p}\{x_n - x_{n+1}\} \leq \sum \int\limits_{x_{n+1}}^{x_n} t^{-p} dt < \infty$. Also

$\int f_1 d\mu = \sum x_n^{-1}\{x_n - x_{n+1}\} = \infty$ since x_n decreases to 0. [See Problem 292 below]. Since $f_2 \leq f_1$ it follows that $\int f_2^p d\mu < \infty$ for $0 < p < 1$. We claim that $\int f_2 d\mu < \infty$. This follows from the fact that $\sum \frac{1}{x_n[1+\log^2 x_n]}(x_n - x_{n+1}) \leq \sum \int\limits_{x_{n+1}}^{x_n} \frac{1}{x[1+\log^2 x]}dx = \int\limits_0^{x_1} \frac{1}{x[1+\log^2 x]}dx \leq \arctan(\log x_1) + \pi/2 < \infty$. Finally we construct f_2, f_3 as follows: there exist disjoint sets B_n with $1 \leq \mu(B_n) < \infty$. [Let $\mu(A_{n+1}) > \mu(A_1) + \mu(A_2) + ... + \mu(A_n) + 1$ and $B_n = A_n \backslash \{A_1 \cup A_2 \cup ... \cup A_{n-1}\}$]. Let $y_n = \mu(B_1) + \mu(B_2) + ... + \mu(B_n)$ and $f_3 = \sum \frac{1}{y_{n+1}} I_{B_{n+1}}, f_4 = \sum \frac{1}{y_{n+1}[1+\log^2 y_{n+1}]} I_{B_{n+1}}$. It can be shown that $\{p \in (0,\infty) : \int |f_3|^p \, d\mu < \infty\} = (1,\infty)$ and $\{p \in (0,\infty) : \int |f_4|^p \, d\mu < \infty\} = [1,\infty)$. Thus 2) implies 3). The fact that 1) and 2) are equivalent follows by Problems 289 and 290 above. 3) implies 1) is straightforward.

[Problems 289, 290 and 291 are due to Villani]

Problem 292

If $\{x_n\}$ decreases strictly to 0 then $\sum \frac{1}{x_n}(x_n - x_{n+1}) = \infty$. If $\{x_n\}$ increases strictly to ∞ then $\sum \frac{1}{x_{n+1}}(x_{n+1} - x_n) = \infty$

Let $\{x_n\}$ decrease strictly to 0. If $\sum \frac{1}{x_n}(x_n - x_{n+1}) < \infty$ then $\frac{x_{n+1}}{x_n} \to 1$ and $\frac{1}{x_n}(x_n - x_{n+1}) \geq \frac{1}{2x_{n+1}}(x_n - x_{n+1})$. Also $\sum \frac{1}{x_{n+1}}(x_n - x_{n+1}) \geq \sum \int\limits_{x_{n+1}}^{x_n} \frac{1}{x} dx = \int\limits_0^{x_1} \frac{1}{x} dx = \infty$. The second part follows by replacing $\{x_n\}$ by $\{x_n^{-1}\}$.

Problem 293

Let X be a compact Hausdorf space. Show that $C(X)$ is finite dimensional if and only if X is a finite set.

If X is finite then $C(X) \subset \mathbb{C}^X$ (or $\mathbb{R}^X$ if we are considering real valued continuous functions) which is finite dimensional. Suppse $C(X)$ is finite dimensional. Then so is its algebraic dual. $\{x_n\}$ is a sequence of distinct points then it follows by Urysohn's Lemma that $f \to f(x_n), n = 1, 2, ...$ are a linearly independent in this dual space.

Problem 294

Let $\{a_n\}$ be a bounded sequence of complex numbers and $0 < p < \infty$. Use an elementary argument to show that $\frac{1}{n}\sum_{k=0}^{n-1}|a_k| \to 0$ if and only if $\frac{1}{n}\sum_{k=0}^{n-1}|a_k|^p \to 0$. What conclusions can be drawn if $\{a_n\}$ is not assumed to be bounded?

[See also Problem 277 above].

Let $S = \{p \in (0,\infty) : \frac{1}{n}\sum_{k=0}^{n-1}|a_k|^p \to 0\}$. It suffices to show that S has the following properties:

a) $p \in S, q > p \Rightarrow q \in S$

b) $p \in S \Rightarrow p/2 \in S$

Proof of a) is obvious: $|a_k|^q \leq C^{q-p}|a_k|^p$ where $C = \sup\{|a_k| : k \in \mathbb{N}\}$

b) follows by Cauchy Schwartz inequality: $\frac{1}{n}\sum_{k=0}^{n-1}|a_k|^{p/2} \leq \frac{1}{n}\{\sum_{k=0}^{n-1}1\}^{1/2}\{\sum_{k=0}^{n-1}|a_k|^p\}^{1/2} = \{\frac{1}{n}\sum_{k=0}^{n-1}|a_k|^p\}^{1/2}$

Now let $\{a_n\}$ be arbitrary. An application of Holder's inequality (in place of Cauchy's inequality) shows that $p \in S, q < p \Rightarrow q \in S$. Thus, if $A \neq \emptyset$ then $A = (0,\alpha)$ or $(0,\alpha]$ for some $\alpha > 0$ unless $A = (0,\infty)$. If $a_n = 2^{k-1}$ if $n = 2^k$ and 0 if n is not a power of 2 then $A = (0,1)$. If $a_n = \frac{2^{k-1}}{k}$ if $n = 2^k$ and 0 if n is not a power of 2 then $A = (0,1]$. [$\frac{1+(\frac{2}{1})^p+\ldots+(\frac{2^{k-1}}{k})^p}{2^k} \geq \frac{1}{k^p}\frac{1+2^p+\ldots+(2^{k-1})^p}{2^k} = \frac{1}{k^p}\frac{2^{kp}-1}{2^k(2^p-1)} \to \infty$ as $k \to \infty$ if $p > 1$ so $A \subset (0,1]$. The reverse inclusion is easy]. Replacing $\{a_n\}$ by a suitable power we get examples where $A = (0,\alpha)$ and those where $A = (0,\alpha]$ for any given α. Thus, in all cases $A = \emptyset, A = (0,\infty)$, $A = (0,\alpha)$ for some $\alpha \in (0,\infty)$ or $A = (0,\alpha]$ for some $\alpha \in (0,\infty)$.

Remark : if f is a bounded measurable function on $(0,\infty)$, $\{\mu_n\}$ is a sequence of probability measures on $(0.\infty)$ and $p \in (0,\infty)$ then $\int |f|\, d\mu_n \to 0$ if and only if $\int |f|^p\, d\mu_n \to 0$. This equivalence fails if f is just locally integrable. (Special case: μ_n is the normalized Lebesgue measure on $(0,n)$).

Problem 295

- Let $(\Omega, \mathcal{F}, \mu)$ be a non-atomic finite measure space. [Non-atomic means there are no atoms,i.e. there are no sets $A \in \mathcal{F}$ such that $\mu(A) > 0$ and every set $B \in \mathcal{F}$ which is contained in A satisfies the property $\mu(B) = 0$ or $\mu(B) = \mu(A)$].

Prove Sak's Theorem that there exists a sequence of sets whose measures decrease strictly to 0. Use this to prove the stronger result that $0 < a < \mu(A)$ implies there is a measurable subset B of A with $\mu(B) = a$.

[See also Problem 188 above]

For the first part we have to show that $r \equiv \inf\{\mu(A) : \mu(A) > 0\} = 0$. If possible let $r > 0$. There exists $B \subset \Omega$ with $0 < \mu(B) < \mu(\Omega)$ and this implies $\mu(B) \geq r$ and $\mu(A \backslash B) \geq r$. Let $E_1 = \Omega \backslash B$. Now repeat the argument with B in place of Ω. [Note that the measure of any subset of B with non-zero measure is $\geq r$]. We get $C \subset B$ such that $\mu(C) \geq r$ and $\mu(B|C) \geq r$. Let $E_2 = B \backslash C$. An induction argument produces a disjoint sequence of sets $\{E_n\}$ with $\mu(E_n) \geq r$ for each n. Since this contradicts countable additivity we must have $r = 0$ and we have proved Sak's Theorem. [For the second part we use an argument given in stackexchange.com]. We may suppose $A = \Omega$. Let $0 < a < \mu(\Omega)$. We construct a sequence $\{A_n\}$ of disjoint measurable sets such that $\mu(D_n) < a$ for all n where $D_n = A_1 \cup A_2 \cup ... \cup A_n$. Let A_1 be any set with $0 < \mu(A_1) < a$. Suppose we have constructed $A_1, A_2, ..., A_n$. We consider the following classes of sets:

$\mathcal{F}_n = \{C \in \mathcal{F} : C \cap D_n = \emptyset \text{ and } 0 < \mu(C) < a - \mu(D_n)\}$

$\mathcal{G}_n = \{C \in \mathcal{F} : C \cap D_n = \emptyset \text{ and } 1/n < \mu(C) < a - \mu(D_n)\}$. If $\mathcal{G}_n \neq \emptyset$ pick $A_{n+1} \in \mathcal{G}_n$. Otherwise pick any $A_{n+1} \in \mathcal{F}_n$. [By Sak's Theorem it is clear that $\mathcal{F}_n$ is non-empty]. Then $A_1, A_2, ..., A_n, A_{n+1}$ are disjoint and $\mu(A_1 \cup A_2 \cup ... \cup A_{n+1}) \leq \mu(D_n) + \mu(A_{n+1}) < \mu(D_n) + a - \mu(D_n) = a$. Thus we have proved the existence of a sequence $\{A_n\}$ of disjoint sets such that $\mu(A_1 \cup A_2 \cup ... \cup A_n) < a$ for all n. It follows that $\mu(A) \leq a$ where $A = A_1 \cup A_2 \cup ....$ To complete the proof we show that $\mu(A) = a$. Suppose, if possible, $\mu(A) < a$. There exists $B \subset A^c$ such that $0 < \mu(B) < a - \mu(A)$. For n sufficiently large we have $\frac{1}{n} < \mu(B) < a - \mu(A) \leq a - \mu(D_n)$ which proves that $\mathcal{G}_n \neq \emptyset$ and hence $A_{n+1} \in \mathcal{G}_n$. Thus $\frac{1}{n} < \mu(A_{n+1})$ for all n sufficiently large contradicting the fact that $\sum \mu(A_n) < \infty$.

Problem 296

Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $\delta \equiv \inf\{f'(x) : x \in \mathbb{R}\} > 0$. Show that $f(x) = 0$ for some x.

$f(x) - f(0) \geq \delta x$ for all $x > 0$ so $f(x) > 0$ for large positive x. Similarly, $f(0) - f(-x) \geq \delta x$ and $f(-x) \leq f(0) - \delta x < 0$ for large positive x.

Remark: let $g : \mathbb{R} \to (0, \infty)$ be measurable and $\int\limits_0^\infty g(x)dx = \int\limits_{-\infty}^0 g(x)dx = \infty$.

Let f be the indefinite integral of g defined by $f(x) = \int\limits_0^x g(t)dt$ if $x \geq 0$ and

$f(x) = -\int\limits_x^0 g(t)dt$ if $x < 0$. Then $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Since f is continuous its range is all of $\mathbb{R}$. Above result is a special case of this with g replaced by f' and f by $f - f(0)$.

Problem 297

Let $A \subset [0,1]$ have the following property: for any sequence $\{r_n\} \subset (0,\infty)$ there exist intervals $I_1, I_2, ...$ such that $A \subset I_1 \cup I_2 \cup ...$ and $m(I_n) < r_n$ for each n. Show that $\mu(A) = 0$ for every finite continuous positive measure μ on the Borel sigma field of $[0,1]$ and use this to show that the Cantor set C does not have this property.

We first prove an elementary fact about continuous measures. [Def. μ is continuous if $\mu\{x\} = 0$ for each x]. We claim that given $\epsilon > 0$ there exists $\delta > 0$ such that for any interval J of length less than δ we have $\mu(J) < \epsilon$. For each $x \in [0,1]$ there is an open interval I_x centered at x such that $\mu(I_x) < \epsilon$. Let J_x be the interval with center x and length one third the length of I_x. Let $\{J_{x_1}, J_{x_2}, ..., J_{x_N}\}$ cover $[0,1]$. Let $\delta = \min\{m(J_{x_k}) : 1 \leq k \leq N\}$. Let J be an interval whose length is less than δ. There exists k such that J intersects J_{x_k}. Let $a \in J \cap J_{x_k}$. Let y be any point of J. Then $|y - a| < \delta$. If J_{x_k} has mid-point c and length α then I_{x_k} has center c and length 3α. Now $|y - c| < \delta + |a - c| \leq \delta + \alpha \leq 2\alpha$ by definition of δ. Hence $y \in I_{x_k}$. Thus $J \subset I_{x_k}$ and $\mu(J) \leq \mu(I_{x_k}) < \epsilon$. This proves the claim. Now we choose positive numbers $\delta_n, n = 1, 2, ...$ such that if J is an interval of length less than δ_n then $\mu(J) < \epsilon/2^n$. By hypthesis there exist intervals $I_1, I_2, ...$ such that $A \subset I_1 \cup I_2 \cup ...$ and $m(I_n) < \delta_n$ for each n. It follows that $\mu(A) \leq \sum \epsilon/2^n = \epsilon$. Since ϵ is arbitrary $\mu(A)$ must be 0.

[Remark: a Borel set has 'strong measure 0' if it has the property above. Borel's conjecture says that a set has strong measure 0 if and only if it is countable].

The Cantor function yields a probability measure μ such that $\mu(C) = 1$. Hence Cantor set does not have strong measure 0.

Problem 298

Let $A \subset \mathbb{R}, \{x_n\} \subset \mathbb{R}$ and assume that $A \backslash U$ is at most countable for any open set U containing $\{x_n\}$. Show that A has strong measure 0. [See Remark in Problem 297 for the definition]

Let $\{r_n\} \subset (0,\infty)$. Let $I_n = (x_n - r_{2n}, x_n + r_{2n})$ and $U = \bigcup\limits_n I_n$. U is open and $\{x_n\} \subset U$. Hence $A \backslash U$ is at most countable, say $A \backslash U \subset \{y_1, y_2, ...\}$. Let $J_n = (y_n - r_{2n-1}, y_n + r_{2n-1})$. Consider the sequence of intervals $J_1, I_1, J_2, I_2, ...$ The diameters of these are $2r_1, 2r_2, 2r_3, 2r_4, ...$ and $A \subset I_1 \cup J_1 \cup I_2 \cup J_2 \cup ....$

Remark: any countable set has the property above and any set with that property is of strong measure 0. Any set with strong measure zero is a null set w.r.t. any continuous measure.

Problem 299

Show that there is a $\sigma-$ finite Borel measure μ on $\mathbb{R}$ such that $\mu((a,b))=\infty$ whenever $a<b$ and $\mu << m$.

Let $\mu(E)=\int\limits_E [\sum\limits_{n=1}^{\infty} \frac{f(x-r_n)}{2^n}]^2 dx$ where $\{r_n\}$ is an enumeration of rational numbers and $f(x)=|x|^{-1/2} e^{-|x|}$ if $x\neq 0, f(0)=0$. Since $f\in L^1(\mathbb{R})$, $\sum\limits_{n=1}^{\infty} \frac{f(x-r_n)}{2^n}\in L^1(\mathbb{R})$ and hence the series $\sum\limits_{n=1}^{\infty} \frac{f(x-r_n)}{2^n}$ converges a.e.. Hence μ is a measure and it is absolutely continuous w.r.t. Lebesgue measure. $\mu((a,b))\geq \int\limits_a^b [\frac{f(x-r_n)}{2^n}]^2 dx=\frac{1}{2^{2n}}\int\limits_{a-r_n}^{b-r_n} f^2(x)dx=\infty$ if n is choesn such that $0\in(a-r_n,b-r_n)$ (i.e. $r_n\in(a,b)$). Note that if $d\nu=\phi dx$ where ϕ is a non-negative finite valued measurable function then ν is necessarily $\sigma-$ finite. [$\nu((-N,N)\cap\phi^{-1}[0,N))<\infty$ for each N].

Problem 300 [Order structure of positive finite mesaures]

Let λ and μ be finite positive measures and $\nu(A)=\inf\{\int g d\lambda+\int(I_A-g)d\mu :$ $0\leq g\leq I_A\}$. Show that ν is a finite positive measure, $\nu\leq\lambda,\nu\leq\lambda$ and if τ is a finite positive measure with $\tau\leq\lambda,\tau\leq\mu$ then $\tau\leq\nu$. [i.e. $\nu=\min\{\lambda,\mu\}$].

We have $\nu(A)\leq\int I_A d\lambda+\int(I_A-I_A)d\mu=\lambda(A)$ and $\nu(A)\leq\int(I_A-I_A)d\lambda+\int I_A d\mu=\mu(A)$ for all A. If we show that ν is finitely additive countable additivity would follow from the fact that $\nu<<(\lambda+\mu)$. Also if $\tau\leq\lambda,\tau\leq\mu$ then $\int g d\lambda+\int(I_A-g)d\mu\geq\int\tau d\lambda+\int(I_A-g)d\tau=\tau(A)$ whenever $0\leq g\leq I_A$ so $\nu(A)\geq\tau(A)$ for all A. Thus it remains only to prove finite additivity of ν. We first note that if A and B are disjoint then $\{g:0\leq g\leq I_{A\cup B}\}=\{g_1+g_2:$ $0\leq g_1\leq I_A, 0\leq g_2\leq I_B\}$. Hence, whenever $0\leq g_1\leq I_A, 0\leq g_2\leq I_B$ we have $\nu(A\cup B)\leq\int(g_1+g_2)d\lambda+\int(I_{A\cup B}-(g_1+g_2))d\mu=\int g_1 d\lambda+\int(I_A-g_1)d\mu+\int g_2 d\lambda+\int(I_B-g_2)d\mu$. Taking infimum over g_1 and g_2 we get $\nu(A\cup B)\leq\nu(A)+\nu(B)$. On the other hand if $0\leq g\leq I_{A\cup B}$ then $g=g_1+g_2$ with $0\leq g_1\leq I_A, 0\leq g_2\leq I_B$ so $\nu(A)+\nu(B)\leq\int g_1 d\lambda+\int(I_A-g_1)d\mu+\int g_2 d\lambda+\int(I_B-g_2)d\mu=\int(g_1+g_2)d\lambda+\int(I_{A\cup B}-(g_1+g_2))d\mu$

$= \int g d\lambda + \int (I_{A\cup B} - g) d\mu$. Taking infimum over g we get $\nu(A) + \nu(B) \leq \nu(A \cup B)$.

Problem 301

Given a sequence $\{\lambda_n\}$ of finite positive measures on a sigma algebra show that there is a largest measure ν such that $\nu \leq \lambda_n$ for all n.

BY Problem 300 we can define $\nu_n = min\{\lambda_1, \lambda_2, ..., \lambda_n\}$. Clearly $\nu(A) = \lim \nu_n(A)$ exists for all A and ν is a measure by Vitali-Hahn-Saks Theorem. [See problem 614]

Remark: if $\lambda_n \leq \lambda$ for some finite positive measure λ then we can apply this result to $\{\lambda - \lambda_n\}$ to show that there is smallest ν with $\lambda_n \leq \nu$ for all n. If λ_n is the restiction of Lebesgue measure to $(-n, n)$ then there is no finte positive ν such that $\lambda_n \leq \nu$ for all n.

Problem 302

Let $\mathcal{C}$ be a family of subsets of Ω which is closed under finite intersections such that $A \in \mathcal{C}$ implies A^c is a finite disjoint union of sets from $\mathcal{C}$. Show that the class of finite unions of sets from $\mathcal{C}$ coincides with the class of finite disjoint unions of sets from $\mathcal{C}$ which coincides with the field generated by $\mathcal{C}$. Hence show that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are fields of subsets of Ω then the field generated by their union is precisely the class of all finite (disjoint) unions of sets of the type $A \cap B$ with $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$.

The second part follows easily from the first (with $\mathcal{C} = \{A \cap B : A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2\}$ since $(A \cap B)^c = (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c)$. We now prove the first part. Let $A \in \mathcal{C}$ and $B \in \mathcal{C}$. Then $A \cup B = A \cup (B \cap A^c) = A \cup (B \cap \{B_1 \cup B_2 \cup ... \cup B_n\}\})$ where $B_i's$ are disjoint, are contained in A^c and belong to $\mathcal{C}$. Thus $A \cup B$ is the union of the disjoint sets $A, B \cap B_1, ..., B \cap B_n$. Assume that union of any k sets in $\mathcal{C}$ is a disjoint union of sets from $\mathcal{C}$ for $1 \leq k \leq N$. Let $A_1, A_2, .., A_{N+!} \in \mathcal{C}$. Consider $A_1 \cup A_2 \cup ... \cup A_{N+1}$. By induction hypothesis we can write $A_2 \cup A_3 \cup ... \cup A_{N+1}$ as a disjoint union $B_1 \cup B_2 \cup ... \cup B_m$ of sets from $\mathcal{C}$. Hence $A_1 \cup A_2 \cup ... \cup A_{N+1} = A_1 \cup B_1 \cup B_2 \cup ... \cup B_m = A_1 \cup (B_1 \backslash A_1) \cup (B_2 \backslash A_1) \cup ... \cup (B_m \backslash A_1)$. Each of the sets $B_i \backslash A_1 = B_i \cap A_1^c, 1 \leq i \leq m$ is a finite disjoint union of members of $\mathcal{C}$. It follows that $A_1 \cup (B_1 \backslash A_1) \cup (B_2 \backslash A_1) \cup ... \cup (B_m \backslash A_1)$ is a finite disjoint union of members of $\mathcal{C}$. The induction argument is now complete. To show that the class of finite unions of sets from $\mathcal{C}$ coincides with the field generated by $\mathcal{C}$ we only have to show that this class is closed under complementation. [For then this class would be a field containing $\mathcal{C}$]. Consider $(A_1 \cup A_2 \cup ... \cup A_N)^c = A_1^c \cap A_2^c \cap ... \cap A_N^c = \bigcap_{i=1}^{N} \bigcup_{j=1}^{j(i)} B_{i,j}$ with $B_{i,j} \in \mathcal{C}$ for all i, j.

We can write $\bigcap_{i=1}^{N} \bigcup_{j=1}^{j(i)} B_{i,j}$ as $\bigcup_{j_1,j_2,...,j_N} (B_{1,j_i} \cap B_{1,j_2} \cap ... \cap B_{1,j_N})$. This completes the proof.

Problem 303

Let $T : L^1(\Omega, \mathcal{F}, P) \to L^1(\Omega, \mathcal{F}, P)$ be linear and $Tf \geq 0$ whenever $f \geq 0$. Show that T is continuous.

[If not there exists $\{f_n\} \subset L^1(\Omega, \mathcal{F}, P)$ such that $\|f_n\| = 1$ and $\|Tf_n\| > 2^n$. Let $f = \sum_{n=1}^{\infty} \frac{|f_n|}{2^n}$. Clearly $f \in L^1(\Omega, \mathcal{F}, P)$. Actually we can take $f_n's$ to be non-negative: T maps real functions to real functions and if $\|Tf\| \leq C \|f\|$ for positive f then this holds for real f, hence complex f with a possibly larger constant C. In this case $Tf \geq T \sum_{n=1}^{N} \frac{f_n}{2^n} > \sum_{n=1}^{N} 1 = N$ for each N].

Remarks: note that under above hypothesis T is also a bounded operator on L^{∞}. There is a converse: if T maps $L^1(\Omega, \mathcal{F}, P)$ into itself boundedly and if it maps $L^{\infty}(\Omega, \mathcal{F}, P)$ into itself boundedly then there is a map $S : L^1(\Omega, \mathcal{F}, P) \to L^1(\Omega, \mathcal{F}, P)$ with $Sf \geq 0$ whenever $f \geq 0$ and $|T^n f| \leq S^n |f|$ for each positive integer n. [Ref.: Lemma 4, page 672 of Dunford and Schwartz "Linear Operators" Part I].

Problem 304

Let X be a compact metric space. If every pointwise convergent sequence in $C(X)$ converges uniformly show that X is a finite set.

There exists a sequence $\{x_n\}$ of distinct points converging to a point x. For each n there exists $\delta_n \in (0, \frac{1}{n})$ such that $x_j \notin B(x_n, \delta_n)$ for all $j \neq n$ and $x \notin B(x_n, \delta_n)$. [If $\delta_n$is small enough there is an open ball $B(x, r)$ which is disjoint from $B(x_n, \delta_n)$ and this ball contains x_j for all j sufficiently large. Hence $x_j \notin B(x_n, \delta_n)$ for all j sufficiently large. Now reduce δ_n further to make sure that no $x_j, j \neq n$ is in $B(x_n, \delta_n)$]. Let $f_n : X \to [0,1]$ be a continuous function which is 1 on $B(x_n, \delta_n/2)$ and 0 on $X \backslash B(x_n, \delta_n)$. Note that $f_n(x_n) = 1$. We claim that $f_n(y) \to 0$ as $n \to \infty$ for every y. If $y \in B(x_n, \delta_n)$ for infinitely many n then $y = \lim x_n = x$ and so $y \notin B(x_n, \delta_n)$ for any n, a contradiction! Thus $y \notin B(x_n, \delta_n)$ for n sufficiently large which implies $f_n(y) = 0$ for n sufficiently large. Hence $f_n \to 0$ pointwise. If $f_n \to 0$ uniformly then $1 = f_n(x_n) \to 0$ a contradiction.

Problem 305

Let X be a normed linear space and T, S be commuting bounded operators on X. Show that TS is invertible if and only if both T and S are invertible. [Invertible operators are those which are bijective and have a bounded inverse].

If X is complete then one could use the fact that if TS is bijective then T and S are both bijective (because $TS = ST$) and boundedness of inverse operators follows by open mapping theorem. The problem is, in fact, purely algebraic! Let A be an algebra with a unit element e over $\mathbb{R}$ or $\mathbb{C}$ and $x, y \in A$ with $xy = yx =$(z,say). If x and y are invertible it is trivial to check that xy is invertible. Suppose $v = z^{-1}$ exists. Then $vy = yv$ as seen by multiplying on both sides of the equation $(xy)y = y(xy)$ by v. Now $(vy)x = vz = e$ and $x(yv) = zv = e$ proving that $vy = yv$ is an inverse of x. Similarly $vx = xv$ is an inverse of y.

Problem 306

[Vey few connected subsets implies lots of clopen sets!]

Let X be a locally compact Hausdorff space. Suppose connected subsets of X are all singleton sets. Show that there is a basis consisting of clopen sets. [clopen means closed and open].

Let $a \in X$ and U be a neighbourhood of a. There exists an open set V such that $a \in V \subset \bar{V} \subset U$ and $\bar{V}$ is compact. We have to show that there is a clopen set containing a and contained in V.

Fact 1: suppose C is closed, $C \subset \bar{V}, b \in \bar{V}$ and, for every $x \in C$ there exists a clopen set W in $\bar{V}$ with $x \in W$ but $b \notin W$.

Then there is a clopen set S of $\bar{V}$ such that $b \in \bar{V} \backslash S$ and $C \subset S$.

To prove this fact let $x \in C$ and pick a clopen set W_x with $x \in W_x$ but $b \notin W_x$. By compactness of $\bar{V}$ (which implies compactness of C) we have $C \subset W_{x_1} \cup W_{x_2} \cup ... \cup W_{x_n}$ for some $\{x_1, x_2, ..., x_n\}$. Take $S = W_{x_1} \cup W_{x_2} \cup ... \cup W_{x_n}$.

Fact 2: let $M = \{x \in \bar{V} : W$ clopen on $\bar{V}$ and $x \in W$ imply $a \in W\}$. Then M is connected.

Assuming Fact 2 we complete the proof as follows: by hypothesis M must be a singleton; since $a \in M$ it follows that $M = \{a\}$. Hence $x \in \bar{V}, x \neq a$ implies there exists a clopen set W containing x which does not contain a. We now apply Fact 1 with $C = \bar{V} \backslash V$ and $b = a$. It follows that there is a clopen set S such that $a \in \bar{V} \backslash S$ and $C \subset S \subset \bar{V}$. Now $T = \bar{V} \backslash S$ is a clopen set in $\bar{V}, a \in T$ and $T \subset \bar{V} \backslash C \subset V$. Since T is closed in $\bar{V}$ it is closed in X. Since T is open in $\bar{V}$ there is an open set U_0 in X such that $T = \bar{V} \cap U_0$. Since $T \subset V$ we get $T = V \cap U_0$. Hence T is also open in X.

It remains to prove Fact 2. Suppose, if possible, $M = A \cup B$ where A and B are disjoint non-empty closed subsets of M. Without loss of generality assume that $a \in A$. It is obvious from the definition of M that $\bar{V} \backslash M$ is open in $\bar{V}$ and hence M is closed in $\bar{V}$. Thus M is closed in X and so are A and B. Since $\bar{V}$ is normal there exist disjoint open sets U_1, U_2 in $\bar{V}$ with $A \subset U_1, B \subset U_2$. We claim that $\bar{U_1} \cap B = \emptyset$. [All closures are in X]. Just note that the closure of U_1 in $\bar{V}$ is same as $\bar{U_1} \cap \bar{V}$ and that the closure of U_1 in $\bar{V}$ is contained in $U_2^c \cap \bar{V}$ (because the latter is closed in $\bar{V}$ and contains U_1). Thus $\bar{U_1} \cap \bar{V} \subset U_2^c \cap \bar{V}$ and hence $\bar{U_1} \cap B = \bar{U_1} \cap B \cap \bar{V} \subset U_2^c \cap \bar{V} \cap B = \emptyset$ because $B \subset U_2$. Let us denote U_1 by W so that W is open in $\bar{V}$, $\bar{W} \cap B = \emptyset$ and $A \subset W$. Now $(\bar{W} \backslash W) \cap M = \varnothing$ because $(\bar{W} \backslash W) \cap A = \varnothing$ and $(\bar{W} \backslash W) \cap B = \varnothing$. Hence, any point z of $(\bar{W} \backslash W) \cap \bar{V}$ belongs to a clopen set S_z in $\bar{V}$ which does not contain a. We now apply Fact 1 to the closed set $(\bar{W} \backslash W) \cap \bar{V}$ of $\bar{V}$. [This set is closed in $\bar{V}$ because W is open in $\bar{V}$]. Thus there exists a clopen set T in $\bar{V}$ such that $a \in \bar{V} \backslash T$ and $(\bar{W} \backslash W) \cap \bar{V} \subset T$. Now consider $H = (W \backslash T) \cap \bar{V}$. This set is open in $\bar{V}$. Since $(\bar{W} \backslash W) \cap \bar{V} \subset T$ we have $H = (\bar{W} \backslash T) \cap \bar{V}$. Hence H is also closed in $\bar{V}$. Note that $a \in H$ and $H \cap B = \emptyset$. Finally we note that $\bar{V} \backslash H$ is a clopen subset of $\bar{V}$ which does not contain a but contains all points of B. Since B is non-empty there is a point u in $B \subset M$ which has a clopen (in $\bar{V}$) neighbourhood (viz. $\bar{V} \backslash H$) not containing a. This contradicts the definition of M. The proof is now complete.

Problem 307

Prove that there is an enumeration $\{q_1, q_2, ...\}$ of $\mathbb{Q} \cap (0,1)$ such that $\sum_{n=1}^{\infty} q_1 q_2 ... q_n = \infty$.

Let $b_n = \frac{n}{n+1}$ and $\{a_1, a_2, ...\}$ is an enumeration of $\mathbb{Q} \cap (0,1) \backslash \{b_1, b_2, ...\}$. We take $\{q_1, q_2, ...\}$ to be $\{a_1, b_1, b_2, ..., b_{n_1}, a_2, b_{n_1+1}, b_{n_1+2}, ..., bn_2, a_3, b_{n_2+1}, ...\}$ for a suitable sequence $\{n_k\}$ of positive integers tending to ∞. Consider $\sum_{n=1}^{n_1+1} q_1 q_2 ... q_n = a_1 + a_1 b_1 + a_1 b_1 b_2 + ... + a_1 b_1 b_2 ... b_{n_1} = a_1 [1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n_1+1}]$. We can choose n_1 such that this last expression exceeds 1. We can then choose n_2 such that $\sum_{n=n_1+1}^{n_2} q_1 q_2 ... q_n > 2$ and so on. In general we can make $\sum_{n=n_j+1}^{n_{j+1}} q_1 q_2 ... q_n > j$.

Problem 308

Show that there is an enumeration $\{r_n\}$ of rationals such that $\mathbb{R} \neq \bigcup_n (r_n - \frac{1}{n}, r_n + \frac{1}{n})$.

Enumerate non-square positive integers as an increasing sequence $\{m_1, m_2, ...\}$. Let ρ_n be a rational number such that $|\rho_n - 1| < \frac{1}{m_n}$ with $\rho'_n s$ distinct and enumerate $\mathbb{Q} \backslash \{\rho_1, \rho_2, ...\}$ as $\{s_1, s_2, ...\}$. Let $r_{n^2} = s_n$ and $\{r_n : n \text{ non-square}\}$ be an the sequence $\rho_1, \rho_2, ...$ Clearly, $\{r_n\}$ is an enumeration of all rationals. Consider $(r_n - \frac{1}{n}, r_n + \frac{1}{n})$ where n is non-square. By definition $r_n = \rho_{k_n}$ for some k_n and $k_n \geq n$. If $|x - r_n| < 1/n$ then $|x - 1| < \frac{1}{n} + \frac{1}{m_{k_n}} \leq \frac{2}{n} \leq 2$. It follows that the interval $(r_n - \frac{1}{n}, r_n + \frac{1}{n})$ is contained in $[-1, 3]$ whenever n is not a square. When $n = m^2$ this interval is $(s_m - \frac{1}{m^2}, s_m + \frac{1}{m^2})$ and Lebesgue measure of the union of all these intervals does not exceed $\sum_m \frac{2}{m^2}$. It follows that the measure of $\bigcup_n (r_n - \frac{1}{n}, r_n + \frac{1}{n})$ is finite.

Problem 309

Let (X, d) be a separable metric space and f a real valued function on X. Show that the set of points x such that $\lim_{y \to x} f(y)$ exists and is different from $f(x)$ is at most countable.

We may suppose that $\lim_{y \to x} f(y)$ exists for every $x \in X$. Let $A = \{x \in X : f(x) < \lim_{y \to x} f(y)\}$. Then $A = \bigcup_{\substack{p,q \in \mathbb{Q} \\ p<q}} A_{p,q}$ where $A_{p,q} = \{x \in X : f(x) < p < q < \lim_{y \to x} f(y)\}$. If we show that $A_{p,q}$ is atmost countable it would follow that A is at most countable. Changing f to $-f$ we can conclude that $\{x \in X : f(x) > \lim_{y \to x} f(y)\}$ is at most countable. Thus, outside a countable set we have $\lim_{y \to x} f(y) \leq f(x) \leq \lim_{y \to x} f(y)$ which means $\lim_{y \to x} f(y) = f(x)$. Now fix p, q and let $x \in A_{p,q}$. If every ball $B(x, r)$ contains a point y of $A_{p,q}$ then there exist points y_n in $B(x, \frac{1}{n}) \cap A_{p,q}, n = 1, 2, ...$ But then $f(y_n) < p$ for all n so $\lim_{y \to x} f(y) \leq p < q$ a contradiction. Thus no point of $A_{p,q}$ is a limit point of $A_{p,q}$ and this implies every sigleton set in $A_{p,q}$ is open in the subspace topology. Since $A_{p,q}$ is separable it must be at most countable.

Problem 310

a) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the equations $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that either $f(x) = x$ for all x or $f(x) = 0$ for all x.

b) Let $g : \mathbb{C} \to \mathbb{C}$ satisfy the equations $g(z_1 + z_2) = g(z_1) + g(z_2)$ and $g(z_1 z_2) = g(z_1)g(z_2)$ for all $z_1, z_2 \in \mathbb{C}$. If g is also continuous show that $g(z) = 0$ for all z or $g(z) = z$ for all z or $g(z) = \bar{z}$ for all z.

c) Determine all multiplicative measurable maps $f : \mathbb{R} \to \mathbb{R}$.

[See also Problem 311 below].

Since $f(rx) = rf(x)$ and $f(rx) = f(r)f(x)$ for r rational we get $f(r) = r$ for r rational. Let $a < b$ and $c = \sqrt{b-a}$. Then $f(b) - f(a) = f(b-a) = f(c^2) = [f(c)]^2 \geq 0$ so f is increasing. If $r_n, s_n (n = 1, 2, ...)$ are rationals with $r_n \uparrow x$ and $s_n \downarrow x$ then $r_n = f(r_n) \leq f(x) \leq f(s_n) = s_n$ for all n so $x \leq f(x) \leq x$. This proves a). Now $g(rz) = rg(z)$ and $g(rz) = g(r)g(z)$ for r rational so $g(r) = r$ for all rational r unless $g \equiv 0$. By continuity $g(x) = x$ for all real x. Now $-1 = g(-1) = [g(i)]^2$ so $g(i) = \pm i$. If $g(i) = i$ then $g(a+ib) = g(a) + g(i)g(b) = a+ib$ for all $a, b \in \mathbb{R}$. On the other hand $g(i) = -i$ gives $g(a+ib) = g(a) + g(i)g(b) = a - ib$ for all $a, b \in \mathbb{R}$. This finishes b). Now let $f : \mathbb{R} \to \mathbb{R}$ be multiplicative. We first observe that $f^2(0) = f(0)$ and $f^2(1) = f(1)$ so $f(0)$ and $f(1)$ both belong to $\{0, 1\}$. If $f(1) = 0$ then $f(x) = f(x)f(1) = 0$ for all x. Now let $f(1) = 0$. If $f(x) = 0$ for some $x \neq 0$ then $f(y) = f(x)f(\frac{y}{x}) = 0$ for all y so assume $f(x) \neq 0$ if $x \neq 0$. Let $g(x) = \log|f(e^x)|$. Then g is additive and measurable so $g(x) = cx$ for some constant c. Thus $|f(e^x)| = e^{cx}$ and $|f(y)| = y^c$ for all $y > 0$. If $x \in \mathbb{R}$ then $f(-e^x) = f(-1)f(e^x)$ so $|f(-e^x)| = |f(-1)|\,e^{cx}$ which gives $|f(-y)| = dy^c$ if $y > 0$. Since $|f(1)| = |f(-1)|\,|f(-1)|$ we get $1^c = d1^c d1^c$ so $d^2 = 1$. Obviously d is positive so $d = 1$. Thus $|f(y)| = |y|^c$ for all $y \in \mathbb{R}$. [$|f(0)| = |f(0)|\,|f(2)| = |f(0)|\,2^c$ so $f(0) = 0$]. Note that $f(x^2) = [f(x)]^2 > 0$ for $x \neq 0$ (because $f(x) \neq 0$ if $x \neq 0$]. Thus $f(y) > 0$ for $y > 0$. If $f(-1) > 0$ then $f(y) > 0$ for all $y < 0$ and we get $f(y) = |y^c|$ for every real number y. Otherwise, $f(-1) < 0$ because $f(x) \neq 0$ if $x \neq 0$ and we get $f(-y) = f(-1)f(y) < 0$ for all $y > 0$. In this case we get $f(y) = |y|^c$ or $-|y|^c$ according as $y \geq 0$ or < 0. The two functions we have arrived at are indeed multiplicative measurable maps.

Remark: if $y \to |y|^c$ is additive then $2^c = 1^c + 1^c$ so $c = 1$ which is a contradiction since $y \to |y|$ is not additive. If $y \to |y|^c\, sgn(y)$ is additive then $c = 1$ again so the only additive and multiplicative map is the identity map. Thus c) contains a).

Problem 311

Find all continuous maps $f : \mathbb{R} \to S^1$ such that $f(x+y) = f(x)f(y)$ for all x, y. Do the same when S^1 is replaced by $\mathbb{C}$.

First part: note that $f(0) = 1$. Fix a positive integer N. By a standard argument in Complex Analysis there exists a unique continuous function $h_N : [-N, N] \to \mathbb{R}$ such that $f(x) = e^{ih_N(x)}$ $(|x| \leq N)$ and $h_N(0) = 1$. It follows easily that $h_N's$ define a continuous function $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$ and $f(x) = e^{ih(x)}$ for all real numbers x. Note that $e^{i[h(a+b)-h(a)-h(b)]} = 1$ so $h(a+b) - h(a) - h(b) = 2n\pi$ for some integer n. By continuity of h we conclude that n does not depend on a and b. Since $h(0) = 0$ we conlude that h is additive.

Since h is additive and continuous there is a real number a such that $h(x) = ax$ for all x. Hence $f(x) = e^{iax}$. Now consider the second part. Since $f(0) = f^2(0)$ either $f(0) = 0$ or $f(0) = 1$. If $f(x) = 0$ for some x then $f(x+y) = f(x)f(y) = 0$ for all y which gives $f \equiv 0$. If this is not the case then $f(0) = 1$ and f never vanishes. Let $g(x) = \frac{f(x)}{|f(x)|}$. The first part can be applied to g and we get $f(x) = e^{iax}\,|f(x)|$. Also $\log|f(x)|$ is an additive continuous function on $\mathbb{R}$, so $|f(x)| = e^{bx}$ for some real number b. We now have $f(x) = e^{(b+ia)x}$.

Problem 312

Find a sequence of continuous functions $\{f_n\}$ from $\mathbb{R}$ into $\mathbb{R}$ such that the sequence converges pointwise to 0 on $\mathbb{R}$ but it does not converge uniformly on any (non-degenerate) interval.

Let $\{r_n\}$ be an enumeration of rationals, $\phi_n(x) = nx$ for $0 \le x \le \frac{1}{n}, \phi_n(x) = n(\frac{2}{n} - x)$ for $\frac{1}{n} \le x \le \frac{2}{n}$ and $\phi_n(x) = 0$ everywhere else. Let $f_n(x) = \sum_{j=1}^{\infty} \frac{\phi_n(x-r_j)}{2^j}$. The fact that $\phi_n(x) \to 0$ as $n \to \infty$ for every real number x implies that $\{f_n\} \to 0$ pointwise. Now let $a < b$. Pick a rational r_j in (a, b). If n is sufficiently large then $r_j + \frac{1}{n} \in (a, b)$ and $f_n(r_j + \frac{1}{n}) \ge \frac{\phi_n(\frac{1}{n})}{2^j} = \frac{1}{2^j}$. It follows that $\{f_n\}$ does not converge uniformly on (a, b).

Problem 313

Let $f \in C[0, 1]$. Suppose $x_1 < x_2 < ... < x_n$ and $y_1, y_2, ..., y_n \in \mathbb{R}$ with no x_i belonging to $[0, 1]$. Given $\epsilon > 0$ show that there is a polynomial p such that $\sup\{|f(x) - p(x)| : 0 \le x \le 1\} < \epsilon$ and $f(x_i) = y_i, 1 \le i \le n$.

There is a polynomial ϕ such that $\phi(x_i) = -y_i$ for $1 \le i \le n$. [For example we can take $\phi(x) = \sum_{i=1}^{n} c_i \prod_{j \ne i}(x - x_j)$ where $c_i = -\frac{y_i}{\prod_{j \ne i}(x_i - x_j)}$ provided $n \ne 1$. For $n = 1$ we can take $\phi(x) = -y_1 + (x - x_1)$]. Now consider $\frac{f(x)+\phi(x)}{\prod_{j=1}^{n}(x-x_j)}$. This function is continuous on $[0, 1]$ and hence there is a polynomial p_0 such that $\left|\frac{f(x)+\phi(x)}{\prod_{j=1}^{n}(x-x_j)} - p_0(x)\right| < \delta$ for $0 \le x \le 1$ where $\delta > 0$ is chosen such that $\delta \sup\{\prod_{j=1}^{n}[1 + |x_j| : 0 \le x \le 1\} < \epsilon$. It follows that $\left|f(x) + \phi(x) - \prod_{j=1}^{n}(x - x_j)p_0(x)\right| < \delta \sup\{\left|\prod_{j=1}^{n}(x - x_j)\right| : 0 \le x \le 1\} \le \delta \sup\{\prod_{j=1}^{n}[1+ |x_j| : 0 \le x \le 1\} < \epsilon$.

Now take $p(x) = \prod_{j=1}^{n}(x - x_j)p_0(x) - \phi(x)$.

Problem 314

Let A be a bounded Borel set set of positive measure . Find all real numbers x such that $x + A$ almost contains A in the sense $m(A\backslash(x + A)) = 0$.

Let $A \subset [a, b]$. If $m(A\backslash(x+A)) = 0$ and $m(A\backslash(y+A)) = 0$ then $m(A\backslash(x+y+A)) = 0$. Hence $m(A\backslash(nx+A)) = 0$ has this property for all $n \in \mathbb{N}$. If $x > 0$ and $nx > b-a$ then $A \cap (nx+A) \subset [a, b] \cap (b, \infty) = \emptyset$ so $m(A\backslash(nx+A)) = m(A) > 0$. This shows that $x \leq 0$. Similarly $x \geq 0$. Thus x must be 0.

Remark: if $0 < m(A) < \infty$ then $\{x : m(A\backslash(x + A)) = 0\}$ is a subgroup of $(\mathbb{R}, +)$.

The next three problems are from Berkely Problem Book.

Problem 315

Let $f : [0, \infty) \to [0, \infty)$ be monotonically increasing. Suppose $f(a) > a$ and $f(b) < b$ for some $a < b$. Show that f has a fixed point.

Replacing f by $f(x + a) - a$ and b by $b - a$ we can reduce the proof to the case $a = 0$. Thus $f(0) > 0$ and $f(b) < b$. We may redefine f to be the constant $f(b)$ on (b, ∞). In this case $f(x) < x$ for all $x \geq b$. Let $A = \{x : f(x) \geq x\}$. Note that $0 \in A$ and $A \subset [0, b)$. Let $\alpha = \sup A$. Note that $\alpha > 0$. If $\epsilon > 0$ we can find $x \in A$ such that $x \leq \alpha < x + \epsilon$. We have $\alpha - f(\alpha) \leq \alpha - f(x)$ (by monotonicity) so $\alpha - f(\alpha) \leq \alpha - f(x) < x + \epsilon - f(x)$. Since $x \in A$ this gives $\alpha - f(\alpha) < \epsilon$. Since ϵ is arbitratry we get $\alpha \leq f(\alpha)$. To complete the proof we have to show that equality holds here. Suppose $\alpha < f(\alpha)$. Let $\delta = f(\alpha) - \alpha$. There exists $y \in A$ with $y \leq \alpha < y+\delta$. We then get $y \leq \alpha < y+\delta = f(\alpha)-\alpha+y$ and hence $f(\alpha) - \alpha + y \leq f(\alpha) \leq f(f(\alpha) - \alpha + y)$ again by monotonicity. We have proved that $z \leq f(z)$ where $z = f(\alpha) - \alpha + y$. Since $f(t) < t$ for all $t \geq b$ we must have $z < b$. Since $\alpha = \sup A$ and $z \in A$ by definition of A we get $z \leq \alpha$ which says $f(\alpha) - \alpha + y \leq \alpha$ or $\delta + y \leq \alpha$. This is a contradiction.

Problem 316

Let $f, f_1, f_2, ... : \mathbb{R} \to \mathbb{R}$. Suppose $f_n(x_n) \to f(x)$ whenever $x_n \to x$. Show that f is continuous.

Remark: $f_n's$ need not be continuous; the proof below works if $\mathbb{R}$ is replaced by any metric space.

If not we have $x_n \to x$ and $|f(x_n) - f(x)| \geq \delta > 0$ for all n. We have $f_m(x_n) \to f(x_n)$ as $m \to \infty$ and so there exists m_n such that $|f_{m_n}(x_n) - f(x_n)| < \delta/2$. We may suppose $m_1 < m_2 < ....$ Also $f_{m_n}(x_n) \to f(x)$. [Indeed

$y_k \to y$ implies $f_{n_j}(y_j) \to f(y)$ for $n_j \uparrow \infty$ as seen by considering a sequence of the type $y_1, y_1, ..., y_1, y_2, y_2, ..., y_2, ...$]. Now $\delta \leq |f(x_n) - f(x)| \leq |f_{m_n}(x_n) - f(x_n)| + |f_{m_n}(x_n) - f(x)| < \delta/2 + |f_{m_n}(x_n) - f(x)|$. Letting $n \to \infty$ we get $\delta \leq \delta/2$, a contradiction.

Problem 317

Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(x) \leq \frac{1}{2\delta} \int\limits_{x-\delta}^{x+\delta} f(t)dt$ or all $\delta > 0$. Show that f is convex.

Let $a < b$ and $g(x) = f(x) - [(x-a)f(b) + (b-x)f(a)]/(b-a)$. Easy to see that g is continuous and $g(x) \leq \frac{1}{2\delta} \int\limits_{x-\delta}^{x+\delta} g(t)dt$ or all $\delta > 0$. Clearly if g attains its maximum at a point $x \in (a,b)$ then g is a constant. Thus g attains its maximum at one of the end points. Since $g(a) = g(b) = 0$ we get $g(x) \leq 0$ for all x which means $f(x) \leq [(x-a)f(b) + (b-x)f(a)]/(b-a)$ we have proved that f is convex.

Problem 318

Let $f \in L^1(a,b)$ where $0 < a < b < 2\pi$. If $\int\limits_a^b f(x)e^{inx}dx = 0$ for all $n \geq 0$ show that $f = 0$ a.e..

This is an easy consequence of some basic theorems in H^p spaces. Extend f to $[0, 2\pi]$ by making it 0 outside (a,b). f is the boundary function of an H^1 function on the unit disc and it vanishes on a set of positive measure (because $b - a < 2\pi$). This implies that $f = 0$ almost everywhere.

Remark: if μ is a complex Borel measure on (a,b) with $\int\limits_{(a,b)} e^{inx}d\mu(x) = 0$ for all $n \geq 0$ then $\mu = 0$. This follows from the fact that μ is absolutely continuous (by F and M Riesz Theorem).

Problem 319

Let X be a normed linear space, M as subpace of finite co-dimension. If M is complete so is X.

There exist $y_1, y_2, .., y_N$ such that any point of X is uniquely expressible as $m + \sum\limits_{i=1}^{N} c_i y_i$ with $m \in M$ and $c_i's$ belonging to the scalar field. Let $\{m_n +$

$\sum_{i=1}^{N} c_{i,n}y_i\}$ be a Cauchy sequence in X. Let $a_n = \sum_{i=1}^{N} |c_{i,n}|$. If $\{a_n\}$ is unbounded then $\{\frac{1}{a_n}(m_n + \sum_{i=1}^{N} c_{i,n}y_i)\} \to 0$ through a subsequence and there is a further subsequence along which $\frac{1}{a_n}(c_{1,n}, c_{2,n}, ..., c_{N,n})$ converges to a unit vector in $\mathbb{R}^N$. This leads to an equation of the type $m + \sum_{i=1}^{N} c_iy_i = 0$ where $(c_1, c_2, ..., c_N)$ is a unit vector. This is a contradiction, so $\{a_n\}$ is bounded. It follows that $\{m_n\}$ is Cauchy along a subsequence (because $\{\sum_{i=1}^{N} c_{i,n}y_i\}$ is convergent, hence Cauchy, along a subsequence). If (along a subsequence) $m_n \to m \in M$ then it follows that $\{m_n + \sum_{i=1}^{N} c_{i,n}y_i\}$ converes along a subsequence, hence along the whole sequence.

Problem 320

Let $\epsilon > 0$. There exists a positive integer N such that $c \in S^1$ and $c, c^2, .., c^N$ have real parts strictly positive imply $|c - 1| < \epsilon$.

Suppose this is false for some $\epsilon > 0$. Then there exists a sequence $\{c_n\}$ in S^1 such that $c_n, c_n^2, .., c_n^n$ have real parts strictly positive but $|c_n - 1| \geq \epsilon$. If c is a limit point of this sequence then $\operatorname{Re} c^k \geq 0$ for every positive integer k and $|c - 1| \geq \epsilon$. If c is not a root of unity then $\{c, c^2, ...\}$ is dense in S^1 and hence $\operatorname{Re} c^{k_j} \to \operatorname{Re}(-1) = -1$ for some $k_j \uparrow \infty$ which is a contradiction. Hence there is a least $N \geq 2$ such that $c^N = 1$. The numbers $c, c^2, ..., c^N$ are distinct and they are all $N - th$ roots of 1. Hence every $N - th$ root of 1 has positive real part. This is a contradiction because $\operatorname{Re} e^{2\pi i \frac{N//2}{N}} < 0$ if N is even and Re $e^{2\pi i \frac{(N-1)//2}{N}} < 0$ if N is odd. [Note that $\pi - \pi/N \in (\pi/2, \pi)$].

Problem 321

Let X be a topological vector space over $\mathbb{R}$ and $\gamma : X \to S^1$ be a continuous map such that $\gamma(x + y) = \gamma(x)\gamma(y)$ for all x, y. Show that there exists $x^* \in X^*$ such that $\gamma(x) = e^{ix^*(x)}$ for all x.

Remark: X^* can be $\{0\}$; in this case the group $(X, +)$ has no continuous character other than 1. [Let X be the space of all bounded Borel measurable functions on $[0, 1]$ with the metric $d(f, g) = \int \frac{|f-g|}{1+|f-g|}$. Then X is a topological vector space. ($f_n \to 0$ in X iff $\not f \to 0$ in measure, w.r.t. Lebesgue measure). We claim that $X^* = \{0\}$. Let $\Phi \in X^*$ and $\epsilon > 0$. There exists $r > 0$ such that $\int \frac{|f|}{1+|f|} < r$ implies $|\Phi(f)| < \epsilon$. Let $f \in X$ be arbitrary and

$f_i = f2^N I_{(\frac{i-1}{2^N},\frac{i}{2^N})}, 1 \leq i \leq 2^N$. Then $\int \frac{|f_i|}{1+|f_i|} = \int_{\frac{i-1}{2^N}}^{\frac{i}{2^N}} \frac{|f|2^N}{1+|f|2^N} \leq \int_{\frac{i-1}{2^N}}^{\frac{i}{2^N}} 1 < r$ whenever $\frac{1}{2^N} < r$. Hence $|\Phi(f_i)| < \epsilon$ for each i. Since $\frac{1}{2^N}\sum_{i=1}^{2^N} f_i = f$ we get $|\Phi(f)| \leq \max\{|\Phi(f_i)| : 1 \leq i \leq 2^N\} \leq \epsilon$. Since ϵ is arbitrary we get $\Phi \equiv 0$].

There is open set U containing 0 such that U is balanced (i.e. $tx \in U$ whenever $x \in U$ and $|t| \leq 1$) and $|\gamma(x) - 1| < 1$ for all $x \in U$. Note that $\gamma(x) \neq -1$ if $x \in U$. Let $f : U \to \mathbb{R}$ be defined by $\gamma(x) = e^{if(x)}$ and $-\pi < f(x) < \pi$. $f(x)$ is nothing but the principle logarithm of $\gamma(x)$. If $x \in X$ there exists $t > 0$ such that $\frac{1}{t}x \in U$. Let $f(x) = tf(\frac{1}{t}x)\ \forall x \in X$. To see that this is well-defined suppose we also have $\frac{1}{s}x \in U$ where $s > 0$. Consider $\{ax : a \in \mathbb{R}\}$. This subgroup is homeomorphic to $(\mathbb{R}, +)$. The restriction of γ to this subgroup is a determined by a character of $(\mathbb{R}, +)$ and hence $\gamma(ax) = e^{ica}$ for some real number c. Now $\gamma(\frac{1}{t}x) = e^{if(\frac{1}{t}x)}$ and so $e^{c/t} = e^{if(\frac{1}{t}x)}$. It follows that $if(\frac{1}{t}x) = ic/t + 2n\pi i$ for some integer n. If $n \neq 0$ then $|ic/t| = \left|if(\frac{1}{t}x) - 2n\pi i\right| \geq 2|n|\pi - \pi \geq \pi$ so $\frac{\pi t}{c}(\frac{1}{t}x) \in U$ (because U is balanced). Hence $\left|\gamma(\frac{\pi}{c}x) - 1\right| < 1$ which means $\left|e^{i\pi} - 1\right| < 1$ which is a contradiction. Hence $if(\frac{1}{t}x) = ic/t$. Similarly $if(\frac{1}{s}x) = ic/s$. It follows that $tf(\frac{1}{t}x) = sf(\frac{1}{s}x) = c$. We have proved that f is well-defined on X. If $x \in U$ then we can take $t = 1$ in the definition, so f defined on X is indeed an extension of f on U. Note that the principle logarithm is continuous on $S^1\backslash\{-1\}$ so f is continuous on U. If we show that f is linear we can conclude that it is continuous on X (because it is continuos at 0). Suppose $x \neq 0$ and $a \in \mathbb{R}\backslash\{0\}$. There exists $t > 0$ such that $\frac{1}{t}x \in U$. Since $\frac{1}{t|a|}(ax) \in U$ we get $f(ax) = t|a| f(\frac{1}{t|a|}(ax))$ and $f(x) = tf(\frac{1}{t}x)$. If $a > 0$ this gives $f(ax) = af(x)$. Noting that $\gamma(-x) = [\gamma(x)]^-$ for $x \in U$ we see that $e^{if(-x)} = e^{-if(x)}$ and $if(-x) = 2\pi im - if(x)$ for some inreger m. Since $|f(-x) + f(x)| < 2\pi$ we get $m = 0$. It follows that $f(-x) = -f(x)$ (for all $x \in U$). From this it follows easily that the equation holds for all $x \in X$. Hence $f(ax) = af(x)$ whenever a and x are non-zero. Since $f(-0) = -f(0)$ we get $f(0) = 0$ and so $f(ax) = af(x)$ for all $a \in \mathbb{R}$ for all $x \in X$. Finally we prove that f is additive: let V be a symmetric neighbourhood of 0 such that $|f(x)| < \frac{\pi}{3}$ for $x \in V$ and $V + V \subset U$. For $x, y \in V$ we have $x + y \in U$ so $\gamma(x+y) = e^{if(x+y)}$ and $\gamma(x)\gamma(y) = e^{if(x)}e^{if(y)}$. Hence $f(x+y) - f(x) - f(y) = 2n\pi$ for some integer n. However $|f(x+y) - f(x) - f(y)| < \pi$ so $n = 0$. Hence $f(x+y) = f(x) + f(y)$ for all $x, y \in V$. If $x, y \in X$ are arbitrary choose $t > 0$ such that $\frac{1}{t}x \in V$, $\frac{1}{t}y \in V$ and $\frac{1}{t}(x+y) \in V$. We get $f(x+y) = tf(\frac{1}{t}(x+y)) = tf(\frac{1}{t}x) + tf(\frac{1}{t}y) = f(x) + f(y)$.

Problem 322

Let X be a locally compact Hausdorff space, Y a Hausdorff space and $f : X \to Y$ a continuous open surjective map. If K is compact in Y there exists C compact in X such that $f(C) = K$.

For each $x \in f^{-1}(K)$ there exists an open set U_x such that $\bar{U_x}$ is compact and $x \in U_x$. K is covered by the open sets $f(U_x), x \in f^{-1}(K)$. Let $\{f(U_{x_i}) : 1 \leq i \leq n\}$ be a finite subcover. Take $C = f^{-1}(K) \cap \bar{U_{x_1}} \cap \bar{U_{x_2}} \cap ... \cap \bar{U_{x_n}}$.

Problem 323 [Manjunath Krishnapur]

Suppose $\{\mathcal{F}_n\}$ is an increasing sequence of sigma algebras on Ω and $\{\mathcal{G}_n\}$ is an decreasing sequence of sigma algebras on Ω such that $\bigcap_n \mathcal{G}_n$ is trivial with respect to a given probability measure P on a sigma algebra $\mathcal{F}$ which contains each $\mathcal{F}_n$. [A sigma algebra is trivial w.r.t. a probability measure P if every set in it has probability 0 or 1]. Let X be a random variable on $(\Omega, \mathcal{F}, P)$ which is measurable w.r.t. $\sigma\{\mathcal{F}_n, \mathcal{G}_n\}$ (the sigma algebra generated by $\mathcal{F}_n \cup \mathcal{G}_n$) for each n. Does it follow that X is measurable w.r.t. the completion of the sigma algebra generated by all the $\mathcal{F}_n's$?

No! Let $\{Y_n\}$ be i.i.d. non-constant random variables, $\mathcal{F}_n = \sigma\{Y_2, Y_3, ..., Y_n\}, \mathcal{G}_n = \sigma\{S_n, S_{n+1}, ...\}$ where $S_n = Y_1 + Y_2 + ... + Y_n$. Let $X = Y_1$. Since $X = S_n - \{Y_2 + Y_3 + ... + Y_n\}$ it follows that X is measurable w.r.t. $\sigma\{\mathcal{F}_n, \mathcal{G}_n\}$ for each n. By Kolomogorov's $0 - 1$ Law $\bigcap_n \mathcal{G}_n$ is trivial. However sigma algebra generated by all the $\mathcal{F}_n's$ is $\sigma\{Y_2, Y_3, ...\}$ and X is independent of this and hence not measurable w.r.t. this sigma field (or its completion) since it is assumed to be non-constant.

Remark: there is a corresponding question about closed subspaces of a HIlbert space: suppose $\{M_n\}$ is an increasing sequence of closed subspaces of a Hilbert space H and $\{N_n\}$ a decreasing sequence of closed subspaces of H with $\bigcap_{n=1}^{\infty} N_n = \{0\}$. Suppose $x \in M_n + N_n$ for each n. Does it follow that x belongs to the closed subspace M generated by $\bigcup_{n=1}^{\infty} M_n$? The answer again is no: let $\{e_n\}$ be an orthonormal basis for $H, M_n = [span\{e_2, e_3, ..., e_n\}]^-$ and $N_n = [span\{s_n, s_{n+1}, ...\}]^-$ where $s_n = e_1 + e_2 + ... + e_n$. Then $e_1 \in M_n + N_n$ for each n. Since e_1 is orthogonal to each M_n it does not belong to M. Suppose $y \in \bigcap_{n=1}^{\infty} N_n$. Notice that $N_n = \{as_n + \sum_{j=n+1}^{\infty} a_j e_j : a \in \mathbb{C}, a_j\mathbb{C}$ for all j and $\sum_{j=n+1}^{\infty} |a_j|^2 < \infty\}$. (If H is a real Hilbert space we can replace $\mathbb{C}$ by $\mathbb{R}$). Note also that $z \in N_n \Rightarrow < z, e_1 >=< z, e_2 >= ... =< z, e_n >$. It follows that $< y, e_j >$ is independent of j and hence $y = 0$.

Problem 324

An infinitely divisible characteristic function (i.d.c.f) may be the product of two characteristic functions not both of which are infinitely divisible.

Next problem gives a stronger result.

Claim: if $a_n \geq 0$ and $\sum_n a_n < \infty$ then $e^{\sum_n a_n(e^{int}-1)}$ is an i.d.c.f. Indeed $e^{\sum_n a_n(e^{int}-1)} = e^{\int (e^{ttx}-1)d\nu(x)}$ where $\nu = \sum_n a_n \delta_n$ (which ν is a Levy mesure). Now let X take the values 0 and 1 with probabilities 2/3 and 1/3 so that its characteristic function is given by $\phi(t) = \frac{2+e^{it}}{3}$. Let Log denote the principle branch of logarithm on $\{\operatorname{Re} z > 0\}$. Then $Log(\frac{2+z}{3})$ is analytic in $\{|z| < 2\}$ (because $\operatorname{Re}(\frac{2+z}{3}) > 0$ there) so we have a power series expansion $Log(\frac{2+z}{3}) = \sum_n b_n z^n$. We can compute the coefficients by taking z in $(0,1)$. Since $\log(\frac{2+t}{3}) = \log\frac{2}{3} + \log(1+t/2) = \log\frac{2}{3} + \sum_n \frac{(-1)^{n+1}(t/2)^n}{n}$ we get $b_n = \frac{(-1)^{n+1}}{n2^n}$ for $n \geq 1$. Now $\phi(t) = e^{Log(\phi(t))} = e^{\sum_n b_n e^{int}} = e^{\sum_n b_n(e^{int}-1)}$. [Note that $\sum_n b_n = Log(\frac{2+1}{3}) = 0$]. If $a_n = b_n^+$ and $c_n = a_n^-$ it follows that $e^{\sum_n a_n(e^{int}-1)}$ and $e^{\sum_n c_n(e^{int}-1)}$ are inifinitely divisible (by the claim) and $e^{\sum_n c_n(e^{int}-1)}\phi(t) = e^{\sum_n a_n(e^{int}-1)}$ which exhibits an inifinitely divisible characteristic function as the product of two characteristic functions one of which, viz. ϕ, is not inifinitely divisible because no non-constant bounded random variable is inifinitely divisible.

Problem 325

Product of two characteristic functions, neither of which is inifinitely divisible can be inifinitely divisible.

Let X take values $-1, 0, 1, 2, ...$ with probabilities $1/6, (5/12), (5/12)(1/2), (5/12)(1/2^2), ...$ Let $\phi(t) = Ee^{itX}$. Then $\phi(t) = e^{-it}/6 + (5/12)\sum_{n=0}^{\infty} \frac{e^{int}}{2^n} = e^{-it}/6 + (5/12)\frac{1}{1-(1/2)e^{it}} = \frac{1}{3}\frac{2+e^{-it}}{2-e^{it}}$. We would like to express $\phi(t)$ in the form $e^{-\sum_{n=0}^{\infty}(\cos(nt)-1)}$. For this consider the function $2+z$ in $\frac{1}{2} < |z| < 2$. Since $\operatorname{Re}(2+z) > 0$ in this region $Log(2+z)$ is well defined. [Log stands for the principle branch of logarithm; note that $Log = \log$ on $(0,\infty)$]. We have a Laurent expansion $Log(2+z) = \sum_{n=-\infty}^{\infty} a_n z^n$ for $\frac{1}{2} < |z| < 2$. The coefficients a_n can be determined from $\log(2+t) = \sum\limits_{n=-\infty}^{\infty} a_n t^n$ for $\frac{1}{2} < t < 2$. We have $\log(2+t) = \log 2 + \sum\limits_{n=1}^{\infty} \frac{1}{n}(\frac{t}{2})^n(-1)^{n+1}$. Hence $a_n = 0$ for $n < 0, a_0 = \log 2, a_n = \frac{1}{n}(\frac{1}{2})^n(-1)^{n+1}, n = 1, 2, ...$ Now $\phi(t) = \frac{1}{3}\frac{2+e^{-it}}{2-e^{it}} = \frac{1}{3}e^{\sum\limits_{n=0}^{\infty} a_n e^{-int}} e^{-\sum\limits_{n=0}^{\infty} a_n e^{int}(-1)^n} = \frac{1}{3}e^{\sum\limits_{n=-\infty}^{\infty} b_n e^{int}}$ where $b_n = \frac{1}{n}(\frac{1}{2})^n$ for $n = 1, 2, ..., b_0 = 0$ and $b_{-n} = \frac{1}{n}(\frac{1}{2})^n(-1)^{n+1}, n = 1, 2, ...$ Now $|\phi(t)|^2 = \frac{1}{9}e^{2\sum\limits_{n=-\infty}^{\infty} b_n \cos(nt)}$. Since $\sum\limits_{n=-\infty}^{\infty} b_n = \log 3$ we may write $|\phi(t)|^2 = e^{2\sum\limits_{n=-\infty}^{\infty} b_n(\cos(nt)-1)} = e^{2\sum\limits_{n=-\infty}^{\infty} b_n(\cos(nt)-1)} = e^{-\int (1-\cos tx)d\nu(x)}$ where $\nu = \sum\limits_{n=1}^{\infty} 2(b_n + b_{-n})\delta_n$. Clearly, $\sum\limits_{n=1}^{\infty} |b_n| < \infty$ so ν is a finite mesure, hence a Levy measure. It follows that $|\phi|^2$ is an inifinitely divisible characteristic function. ϕ itself is not an inifinitely divisible characteristic function. Indeed if U and V are i.i.d. and $U+V$ has the distribution has the same distribution as X then U and V are discrete random variable. If $P\{U = n\} > 0$ for some $n < 0$ then $P\{U+V = 2n\} \geq P\{U = n\}P\{V = n\} = P^2\{U = n\} > 0$ but $P\{X = j\} = 0$ for $j < -1$. It follows that U and V are non-negative random variable which implies that X is non-negative, a contradiction.

Problem 326

Prove that there is a bounded set E in $\mathbb{R}$ of measure 0 such that $E+E$ is not measurable.

Let $\{a_i\}_{i\in I}$ be a maximal linearly independent subset of the Cantor set C over the field $\mathbb{Q}$. Since $C + C = [0, 2]$ the set $\{a_i\}_{i\in I}$ is also a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. Let $E_1 = A + A$ where $A = \{ra_i : i \in I, r \in \mathbb{Q}, 0 \leq r \leq 1\}$. Clearly A has measure 0. Since $E_1 + E_1$ has no interior because every point in it is a linear combination of four elements of $\{a_i\}_{i\in I}$. (If every point of (a, b) is a linear combination of four elements of $\{a_i\}_{i\in I}$ then, for any real number x there exists n such that $\frac{a+b}{2} + \frac{x}{n} \in (a, b)$ which show that x can be written as a linear combination of 8 elements of $\{a_i\}$. However a sum of 9 elements of $\{a_i\}$ cannot be expressed in this form]. Hence E_1 has measure 0 if it is measurable. If it is not measurable we can take $E = A$. If E_1 is measurable we define $\{E_n\}$ by $E_{n+1} = E_n + E_n, n \geq 1$. If the process does not stop (in the sense we get infintely many measurable sets $\{E_n\}$) then $[0, 1] \subset \bigcup\limits_{n=1}^{\infty} E_n$, a contradiction since some E_n must have positive measure, but $E_n + E_n$ does not contain any interval.

Problem 327

If $f:\mathbb{R}\to\mathbb{R}$ is measurable, $g(x)=\sup\{|f(x+y)-f(x)|:y\in\mathbb{R}\}, h(x)=\sup\{|f(x+y)-f(x-y)|:y\in\mathbb{R}\}$ then g is measurable, but h need not be.

Proof: $\sup\{|f(x+y)-f(x)|:y\in\mathbb{R}\}=\max\{\sup\{f(x+y)-f(x):y\in\mathbb{R}\},-\inf\{f(x+y)-f(x):y\in\mathbb{R}\}\}$

$=\max\{\sup\{f(x+y):y\in\mathbb{R}\}-f(x),-\inf\{f(x+y):y\in\mathbb{R}\}+f(x)\}=\max\{a-f(x),f(x)-b\}$ where $a=\sup\{f(y):y\in\mathbb{R}\}$ and $b=-\inf\{f(y):y\in\mathbb{R}\}$. [If $a=\infty$ or $b=\infty$ then $g\equiv\infty$]. Hence g is measurable.

Let E be a subset of $[0,1]$ of measure 0 such that $E+E$ is not measurable. Let $f(x)=I_E-I_{E+2}$. Then $h=I_F$ where $F=\frac{1}{2}(E+E)+1$. Hence h is not measurable.

Problem 328

Given a measurable function $f:[0,1]\to\mathbb{R}$ there is a continuous function g on $[0,1]$ such that $g'=f$ a.e.

This is a theorem of Lusin.

Lemma1

Let $f\in L^1([0,1])$ and $\epsilon>0$. There exists $g\in C[0,1]$ such that $g'=f$ a.e., $g(a)=0=g(b)$ and $|g(x)|\leq\epsilon\ \forall x$.

Proof of Lemma 1: let $h(x)=\int\limits_0^x f(t)dt$. There is a partition $\{0=a_0,a_1,a_2,...,a_n=1\}$ of $[0,1]$ such that the oscillation of h on $[a_i,a_{i+1}]$ does not exceed ϵ for any i. There is continous monotonic function ϕ_i on $[a_i,a_{i+1}]$ such that $\phi_i(a_i)=h(a_i),\phi_i(a_{i+1})=h(a_{i+1})$ and $h'=0$ a.e. [ϕ_i non-decreasing or non-increasing according as $h(a_{i+1})>h(a_i)$ or $h(a_{i+1})\leq h(a_i)$. This follows by applying affine maps to a Cantor function]. Define $\phi:[0,1]\to\mathbb{R}$ by $\phi=\phi_i$ on $[a_i,a_{i+1}],0\leq i<n\}$. Let $g=h-\phi$. Then g is continuous, $g'=h'=f$ a.e., $g(a)=0=g(b)$ [because $\phi(a_0)=h(a_0),\phi(a_n)=h(a_n)$] and $|g(x)|=|h(x)-\phi(x)|=|h(x)-\phi_i(x)|$ on $[a_i,a_{i+1}]$ and $h(x)-\phi_i(x)\leq h(x)-\phi_i(a_i)$

$=h(x)-h(a_i)\leq\epsilon,\phi_i(x)-h(x)\leq\phi_i(a_{i+1})-h(x)=h(a_{i+1})-h(x)\leq\epsilon$ (if ϕ_i is increasing; a similar argument works if it is decreasing).

Lemma 2

Let $f\in L^1([0,1]),\epsilon>0$ and C be a closed subset of $[0,1]$. Then there exists $g\in C[0,1]$ such that $g'=f$ a.e. on $[0,1]\backslash C,g=g'=0$ on C and $|g(x+h)|<\epsilon|h|$ whenever $x\in C$ and $x+h\in[0,1]$.

Let $[0,1]\backslash C$ be the disjoint union of open intervals (a_n,b_n) and choose $\{a_{n,m}\}\subset(a_n,b_n)$ such that $a_{n,m}\downarrow a_n$ as $m\downarrow-\infty$ and $a_{n,m}\uparrow b_n$ as $m\uparrow\infty$. Let $\epsilon_{n,m}=\min\{\frac{\epsilon(a_{n,m}-a_n)}{n+|m|},\frac{\epsilon(b_n-a_{n,m})}{n+|m|}\}$. By Lemma 1 there exists a continuous function g on $\bigcup(a_n,b_n)$ such that $g(a_n)=g(b_n)=0$ for all $n,|g|\leq\epsilon_{nm}$ on $(a_{n,m},a_{n,m+1})$ and $g'=f$ a.e. on $\bigcup(a_n,b_n)$. Define g to be 0 on C. Claim:

$g' = 0$ on C. Suppose $x \in C$ and $x < a_n < b_n$. Then, for any m and any $y \in (a_{n,m}, a_{n,m+1})$ we have $\left|\frac{g(y)-g(x)}{y-x}\right| \leq \frac{\epsilon_{nm}}{a_{n,m}-a_n} \leq \frac{\epsilon}{n+|m|}$. It follows from this that $g'(x+) = 0$. Similarly $g'(x-) = 0$. Hence $g = g' = 0$ on C. It remains to see that $|g(x+h)| < \epsilon\,|h|$ whenever $x \in C$ and $x + h \in [0,1]$. If $h > 0$ we just have to take $y = x + h$ in the inequality $\left|\frac{g(y)-g(x)}{y-x}\right| \leq \frac{\epsilon_{nm}}{a_{n,m}-a_n} \leq \frac{\epsilon}{n+|m|} < \epsilon$. A similar argument holds for $h < 0$.

Proof of Lusin's theorem: we construct closed sets C_n and continuous functions g_n $(n \geq 0)$ such that the following hold:

1) each g_n is a.e. differentiable and $h_n' = f(x)$ $\forall x \in D_n$ where $D_n = C_0 \cup C_1 \cup ... \cup C_n$ and $h_n = g_0 + g_1 + ... + g_n$

2) $g_n \equiv 0$ on D_{n-1}

3) $|g_n(x+h)| \leq \frac{|h|}{2^n}$ if $x \in D_{n-1}$ and $x + h \in [0,1]$

4) $m(D_n^c) < \frac{1}{n}$ for $n \geq 1$.

Once this is done we show that $h = \lim h_n = \displaystyle\sum_{n=0}^{\infty} g_n$ satisfies $h' = f$ a.e.

We begin the construction of C_n and g_n by taking g_0 to be 0 and C_0 to be $\emptyset$. Suppose we have constructed C_n and g_n for $0 \leq n \leq N$. Let E_N be a measurable subset of D_N^c such that $m(D_N^c \backslash E_N) < \frac{1}{N+1}$ and such that f and h_N' are bounded on E_N. [Clearly such a set exists: intersect D_N^c with $|f| < R$ and $|h_N'| < R$ for a sufficiently large R]. By Lemma 2 there exists g_{N+1} such that

a) $g_{N+1}' = f - h_N'$ a.e.

b) $g_{N+1} = g_{N+1}' = 0$ on D_N

c) $|g_{N+1}(x+h)| \leq \frac{|h|}{2^{N+1}}$ if $x \in D_N$ and $x + h \in [0,1]$

Now we choose a closed set $C_{N+1} \subset E_N$ such that $m(D_N^c \backslash C_{N+1}) < \frac{1}{N+1}$ and such that $g_{N+1}' = f - h_N'$ on C_{N+1}.

The construction is over. Let $C = \displaystyle\bigcup_n C_n$ so that $m(C) = 1$. [$m(C^c) \leq m(D_{N+1}^c) = m(D_N^c \backslash C_{N+1}) < \frac{1}{N+1}$ for all N]. We claim that $h'(x) = f(x)$ if $x \in C$. Fix x. Note that there exists N such that $x \in D_n$ for all $n \geq N$. We have $\frac{h(x+t)-h(x)}{t} = \frac{h_N(x+t)-h_N(x)}{t} + \displaystyle\sum_{j=N+1}^{\infty} \frac{g_j(x+t)-g_j(x)}{t}$ and $\left|\frac{h(x+t)-h(x)}{t} - f(x)\right| \leq \left|\frac{h_N(x+t)-h_N(x)}{t} - f(x)\right| + \displaystyle\sum_{j=N+1}^{\infty} \frac{|t|}{2^j}$. Since $h_N'(x) - f(x) = g_{N+1}'(x) = 0$ we are done.

Problem 329

There exists a continuous function g on $[0,1]$ such that g' exists a.e., $g'(x) > 1$ a.e. but g is not increasing on any interval.

Let $\mathbb{Q} = \{r_n\}$ and $f = \displaystyle\sum_{j=1}^{\infty} (n+1)^2 I_{(r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2})}$. Since $\displaystyle\sum_{j=1}^{\infty} m((r_n - \frac{1}{n^2}, r_n +$

$\frac{1}{n^2})) < \infty$ we have $m(\limsup(r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2})) = 0$. Hence f is finite valued a.e. Clearly f is measurable. Hence, by Problem 328 above there exists a continuous function g such that $g' = f$ a.e.. If g is increasing on $[a,b]$ (for some a, b with $a < b$) then $\int_a^b g'(x)dx \leq g(b) - g(a)$ so $\int_a^b f(x)dx < \infty$. Hence $\sum_{j=1}^{\infty}(n+1)^2 m((r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2}) \cap (a,b)) < \infty$. In particular $(n+1)^2 m((r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2}) \cap (a,b)) \to 0$ as $n \to \infty$. In turn, this implies $(r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2})$ is not contained in (a,b) for n sufficiently large. Let $\delta = \frac{b-a}{4}$. If $r_n \in (\frac{a+b}{2} - \delta, \frac{a+b}{2} + \delta)$ then $(r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2}) \subset (\frac{a+b}{2} - \delta - \frac{1}{n^2}, \frac{a+b}{2} + \delta + \frac{1}{n^2})$

$\subset (a,b)$ provided $\frac{1}{n^2} < \frac{b-a}{4}$. It follows that $r_n \notin (\frac{a+b}{2} - \delta, \frac{a+b}{2} + \delta)$ for n sufficlently large, which is absurd.

Problem 330

Let $f : [0,1] \to \mathbb{R}$ be integrable and $g : [0,1] \to [0,1]$ is continuously differentiable. Is $f(g(t))g'(t)$ necessarily integrable?

No. Let $g(t) = c_n(t - \frac{1}{n+1})^2$ on $[\frac{1}{n+1}, x_n], g(t) = c_n(t - \frac{1}{n})^2$ on $[x_n, \frac{1}{n}] n = 1, 2, ...$ where x_n is the mid-point of $[\frac{1}{n+1}, \frac{1}{n}]$ and $c_n = 4(n+1)^2$ and $g(0) = 0$. It is easy to see that g is continuously differentiable, $g(x_n) = \frac{1}{n^2}$ and g is increasing on $[\frac{1}{n+1}, x_n]$. Let $f(x) = x^{-1/2}$ for $x \neq 0, f(0) = 0$. Then $\int_{1/(n+1)}^{x_n} |f(g(t))g'(t)|\, dt = \int_{1/(n+1)}^{x_n} \left|\frac{2c_n}{\sqrt{c_n}}\right| dt = 2\sqrt{c_n}[x_n - \frac{1}{n+1}] = \frac{4(n+1)}{2n(n+1)} = \frac{2}{n}$.

Hence $\int_0^1 |f(g(t))g'(t)|\, dt \geq \sum_n \frac{2}{n} = \infty$.

Problem 331 [Smital's Lemma]

If A is a set of positive measure in $\mathbb{R}$ then $(A + \mathbb{Q})^c$ has measure 0.

[$\mathbb{Q}$ may be replaced by any countable dense set]

Fix a point a such that $\frac{m((a-\delta, a+\delta)\cap A)}{2\delta} \to 1$ as $\delta \to 0$. [By Lebesgue's Theorem almost all points of A have this property]. Let $0 < \alpha < 1$. Choose $\delta > 0$ such that $m((a-r, a+r) \cap A) > 2\alpha r$ whenever $0 < r < \delta$. Let $D = a + \mathbb{Q}$. If $x \in D$ then $x = a + d$ for some $d \in \mathbb{Q}$. Consider $m((x-r, x+r) \cap (A+d))$. Since $(a-r, a+r) \subset (x-r, x+r) - d$ we have $m((x-r, x+r) \cap (A+d)) \geq m(\{(a-r, a+r)+d\} \cap (A+d)) = m((a-r, a+r) \cap A) > 2\alpha r$. Now $m((x-r, x+r) \cap (A+\mathbb{Q})) \geq m((x-r, x+r) \cap (A+d)) \geq 2\alpha r$. Fix a positive integer N. Consider the collection of all intervals of the type $(x-r, x+r)$ with $x \in D$ and $0 < r < \delta$ which are contained in $(-N, N)$. Since D is dense this family is a Vitali cover for $(-N, N)$. Hence there is a disjoint sequence of intervals $(x_n - r_n, x_n + r_n)$

contained in $(-N,N)$ such that $m((-N,N)\backslash\bigcup_n(x_n-r_n,x_n+r_n))=0$ and $m((x_n-r_n,x_n+r_n)\cap(A+\mathbb{Q}))\geq 2\alpha r_n$ for each n. [Kannan and Kruger Advanced Analysis on the Real Line, p.10 OR Diestel and Spalsbury, The Joys of Haar Measure, p. 13]. Now $m((A+\mathbb{Q})\cap(-N,N))=\sum_n m((A+\mathbb{Q})\cap(x_n-r_n,x_n+r_n))\geq\sum_n 2\alpha r_n=\alpha m((-N,N))$. Since $\alpha<1$ is arbitrary we get $m((A+\mathbb{Q})\cap(-N,N))=m((-N,N))$. Thus $m((A+\mathbb{Q})^c\cap(-N,N))=0$ for every N and the proof is complete.

Problem 332

If K_1 and K_2 are compact subsets of $\mathbb{R}$ (and m is the Lebesgue measure) show that $m(K_1)+m(K_2)\leq m(K_1+K_2)$. Is the reverse inequality true? What if we replace compact sets by open sets?

Translate K_2 by $\sup K_1-\inf K_2$. Then the inequality does not change and the proof is therefore reduced to the case $\sup K_1=\inf K_2(=c$, say). In this case $K_1\cup K_2\subset K_1+K_2-c$ (because $c\in K_1\cap K_2$) and $m(K_1)+m(K_2)=m(K_1\cup K_2)$ (because $K_1\backslash\{c\}$ and K_2 are disjoint) so $m(K_1)+m(K_2)\leq m(K_1+K_2-c)=m(K_1+K_2)$. The reverse inequality fails when $K_1=K_2=C$, the Cantor set.

It is not true that $m(U+V)\leq m(U)+m(V)$ for all open sets: there is an open set U such that $C\subset U$ and $m(U)<1/2$. Since $[0,2]=C+C\subset U+U$ we would have $2\leq m(U+U)\leq 2m(U)<1$ a contradiction. It is true that $m(U)+m(V)\leq m(U+V)$ for any two open sets U and V. Let $\epsilon>0$ and choose compact sets H,K such that $H\subset U,K\subset V,m(U)<m(H)+\epsilon$ and $m(V)<m(K)+\epsilon$. Then $m(U)+m(V)<m(H)+m(K)+2\epsilon\leq m(H+K)+2\epsilon\leq m(U+V)+2\epsilon$.

Problem 333

Let μ be a complex Borel measure on $\mathbb{R}$ such that the conditions $f_n\to 0$ a.e. $[m]$, $f_n's$ uniformly bounded and each f_n is continuous imply $\int f_n d\mu\to 0$. Show that μ is absolutely continuous w.r.t. m.

Proof: let $m(A)=0$. Let K be any compact subset of A. Then $I_K\in L^1(m+|\mu|)$ so we can find a sequence $\{f_n\}$ of continuous functions converging to I_K in $L^1(m+|\mu|)$. Some subsequence $\{f_{n_k}\}$ converges to I_K a.e. $[m+|\mu|]$. Let $g_k=\max\{0,\min\{f_{n_k},1\}\}$. Then $\{g_k\}$ converges to I_K a.e. $[m+|\mu|]$. Since $m(K)=0$ the sequence $\{g_k\}$ converges to 0 a.e. $[m]$. By hypothesis $\int g_k d\mu\to 0$. By DCT $\int g_k d\mu\to\int I_K d\mu$. Hence $\mu(K)=0$. Since K is an arbitrary compact subset of A and $|\mu|$ is regular we get $\mu(A)=0$.

Problem 334

Let f be a twice continuously differentiable map from $\mathbb{R}$ to $\mathbb{R}$ such that $\int_{-1}^{1}\int_{-1}^{1} \frac{|f(x)-f(y)|}{|x-y|^2} dxdy < \infty$. Show that f is a constant.

Note that there is a finite constant C such that $|f(x) - f(y) - (x-y)f'(y)| \leq C(x-y)^2 \; \forall x, y \in [-1, 1]$. Hence $\int_{-1}^{1}\int_{-1}^{1} \frac{\left|f(x)-f(y)-(x-y)f'(y)\right|}{|x-y|^2} dxdy < \infty$. It follows from this and the hypothesis that $\int_{-1}^{1}\int_{-1}^{1} \frac{\left|(x-y)f'(y)\right|}{|x-y|^2} dxdy < \infty$. Hence $\int_{-1}^{1} \frac{\left|(x-y)f'(y)\right|}{|x-y|^2} dx < \infty$ for almost all y. This implies $f'(y) = 0$ a.e.. Since f' is continuous the conclusion follows.

Remark: smoothness of f can be replaced by mesurability by approximating f by smooth fucntions and the result can be further generalized by replacing $\mathbb{R}$ by $\mathbb{R}^n$ and $[-1, 1]$ by any ball in $\mathbb{R}^n$.

Problem 335

Recall Vitali-Hahn-Saks Theorem: if a sequence of complex mesures converges set-wise the limit is a measure. Give a counterexample to show that the limit of a sequence of positive mesaures may not be a measure even if it is not identically ∞. [See problem 614]

Let $p_{ij} = 2^{-j}$ if $1 \leq j \leq i$ and $p_{ij} = 1$ if $j > i$. Let $\mu_n(E) = \sum_{j \in E} p_{nj}$. This gives a sequence of $\sigma-$ finite measures on $\mathbb{N}$ with the sigma algebra of all subsets. Claim: $\mu_{n+1}(E) \leq \mu_n(E)$ for all $E \subset \mathbb{N}$. For this it suffices to observe that $p_{(n+1)j} \leq p_{nj}$ for all n, j. If $j > n$ then the inequality holds because $p_{nj} = 1$. If $j \leq n < n+1$ then$p_{(n+1)j} = 2^{-j} = p_{nj}$. Hence $\mu_{n+1} \leq \mu_n$ and $\nu(E) = \lim \mu_n(E)$ exists for all E. Note that $\nu(\{k\}) = \lim \mu_n(\{k\}) = \lim p_{nk} = 2^{-k}$ for each k. But $\nu(\mathbb{N}) = \lim \mu_n(\mathbb{N}) = \infty$ so $\nu(\mathbb{N}) \neq \sum_{k=1}^{\infty} \nu(\{k\})$.

Problem 336

Show that any function $f : [0, 1] \to [0, 1]$ can be expressed as $g \circ h$ where $g : [0, 1] \to [0, 1]$ is Borel measurable and $h : [0, 1] \to [0, 1]$ is Lebesgue measurable.

Let $h(x) = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$ if $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$ with $a_n \in \{0, 1\}$ for each n. h is well defined if we insist that expansions to base 2 are all infinite expansions (we define $h(0)$

to be 0). Claim: h is strictly increasing (hence 1-1). To see this suppose $\sum\limits_{n=1}^{\infty} \frac{a_n}{2^n} < \sum\limits_{n=1}^{\infty} \frac{b_n}{2^n}$. Let k be the least positive integer with $a_k \neq b_k$. If $a_k > b_k$ we get $\frac{1}{2^k} \leq \frac{a_k - b_k}{2^k} = \sum\limits_{n=1}^{k} \frac{a_n}{2^n} - \sum\limits_{n=1}^{k} \frac{b_n}{2^n} = \sum\limits_{n=1}^{\infty} \frac{a_n}{2^n} - \sum\limits_{n=1}^{\infty} \frac{b_n}{2^n} + \sum\limits_{n=k+1}^{\infty} \frac{b_n}{2^n} - \sum\limits_{n=k+1}^{\infty} \frac{a_n}{2^n}$

$< \sum\limits_{n=k+1}^{\infty} \frac{b_n}{2^n} - \sum\limits_{n=k+1}^{\infty} \frac{a_n}{2^n} \leq \sum\limits_{n=k+1}^{\infty} \frac{1}{2^n} (= \frac{1}{2^k})$ since $b_n - a_n \leq 1$ for all n. This implies that equality holds throughout and $a_k - b_k = 1, b_n - a_n = 1$ for all $n > k$. But then $a_n = 0$ and $b_n = 1$ for all $n > k$ and the expansion $\sum\limits_{n=1}^{\infty} \frac{a_n}{2^n}$ is a finite expansion. We have proved that $a_k < b_k$. This implies $\sum\limits_{n=1}^{\infty} \frac{2a_n}{3^n} < \sum\limits_{n=1}^{\infty} \frac{2b_n}{3^n}$ since $\sum\limits_{n=1}^{\infty} \frac{2b_n}{3^n} - \sum\limits_{n=1}^{\infty} \frac{2a_n}{3^n} \geq \frac{1}{3^k} - \sum\limits_{n=k+1}^{\infty} \frac{2}{3^n} = 0$ and equality can hold only if $a_n - b_n = 1$ and so $b_n = 0$ for all $n > k$, a contradiction. If $x \in h([0,1])$ we define $g(x) = f(h^{-1}(x))$. Otherwise we set $g(x) = 0$. Since $h([0,1])$ is contained in the Cantor set, g is 0 a.e. Hence g is Lebesgue measurable. Also $g(h(y)) = f(y)$ for all y so $f = g \circ h$.

Problem 337

Does Dominated Convergence Theorem hold for nets of measurable functions?

No! For any finite set $I \subset [0,1]$ let f_I be a continuous function which is 1 on I, takes values in $[0,1]$ and satisfies the inequality $\int\limits_0^1 f_I(x)dx < \frac{1}{\#(I)}$ ($\#(I)$ is the cardinality of I). Order finite subsets of $[0,1]$ by inclusion. Then $f_I(x) \to 1$ for each x but $\{\int\limits_0^1 f_I(x)dx\} \to 0$.

Problem 338

Show that $X_n \xrightarrow{P} X$ if and only if $Q \circ X_n^{-1} \xrightarrow{w} Q \circ X^{-1}$ for every probability measure Q equivalent to P. [Q equivalent to P means $Q << P$ and $P << Q$]

Proof: let $Y_n = \tan^{-1}(X_n)$ and $Y = \tan^{-1}(X)$. Then $X_n \xrightarrow{P} X$ iff $Y_n \xrightarrow{P} Y$. Also $Q \circ X_n^{-1} \xrightarrow{w} Q \circ X^{-1}$ iff $Q \circ Y_n^{-1} \xrightarrow{w} Q \circ Y^{-1}$. Hence there is no loss of generality in assuming that X and $X_n's$ are uniformly bounded. It is clear that $X_n \xrightarrow{P} X$ implies $X_n \xrightarrow{Q} X$ and hence $Q \circ X_n^{-1} \xrightarrow{w} Q \circ X^{-1}$. We now assume that $Q \circ X_n^{-1} \xrightarrow{w} Q \circ X^{-1}$ whenever Q is equivalent to P. We have to show convergence in probability. Let $P(A) > 0, 0 < t < 1$ and $Q(B) = tP(A \cap B) + (1-t)P(A^c \cap B)$. Then Q is a probability measure equivalent to P. Hence $Q \circ X_n^{-1} \xrightarrow{w} Q \circ X^{-1}$ which implies $\int X_n dQ \to \int X dQ$. Given $\epsilon > 0$ choose t such that $\frac{1-t}{t} < \epsilon/(2M)$ and $t > 1/2$ where M is an upper bound for $|X|$ and $|X_n|' s$. Since $\left|\int X_n dQ - \int X dQ\right| < \epsilon$ for n sufficiently large we get $\left|t\int\limits_A X_n dP + (1-t)\int\limits_{A^c} X_n dP - t\int\limits_A X dP - (1-t)\int\limits_{A^c} X dP\right| < \epsilon$ for such n. But then $\left|t\int\limits_A X_n dP - t\int\limits_A X dP\right| < \epsilon + \left|(1-t)\int\limits_{A^c} X_n dP - (1-t)\int\limits_{A^c} X dP\right| < \epsilon + 2M(1-t)$ which gives $\left|\int\limits_A X_n dP - \int\limits_A X dP\right| < \frac{\epsilon}{t} + 2M\frac{1-t}{t}$

$< 2\epsilon + \epsilon = 3\epsilon$. We have proved that $\int\limits_A X_n dP \to \int\limits_A X dP$ for each A. This implies that $\int Z X_n dP \to \int Z X_n dP$ for any simple function $Z \in L^2(P)$ hence for all $Z \in L^2(P)$. If we show that $\int X_n^2 dP \to \int X^2 dP$ it would follow that $X_n \to X$ in $L^2(P)$ and hence in probability. [If a sequence $\{x_n\}$ in a Hilbert space converges weakly to x and if $\|x_n\| \to \|x\|$ then $\|x_n - x\| \to 0$. Since $Q \circ X_n^{-1} \xrightarrow{w} Q \circ X^{-1}$ for every probability measure Q equivalent to P we have $P \circ X_n^{-1} \xrightarrow{w} P \circ X^{-1}$ and so $\int X_n^2 dP \to \int X^2 dP$.

Problem 339

Let (X, d) be a connected separable matric space and assume that X is not a finite set. Show that there is a measure μ on the Borel subsets of X such that $\mu(U) = \infty$ for every non-empty open set U but μ is $\sigma-$ finite.

Connectedness is used only to assert that there are no isolated points. Let $\{x_n\}$ be a countable dense set (with $x_n's$ distinct) and $\mu = \sum\limits_{n=1}^{\infty} \delta_{x_n}$. If $A_n = \{x_{n+1}, x_{n+2}, ...\}$ then $\cap A_n = \emptyset$ and $\cup A_n^c = X$. Since $\mu(A_n^c) = N < \infty$ it follows that μ is $\sigma-$ finite. Suppose U is a non-empty open set and $\mu(U) < \infty$. Then U contains at most finite number of $x_n's$. There is a smaller non-empty open set containing a single point x_{n_0} of $\{x_1, x_2, ...\}$. If this smaller open set contains a point x other than x_{n_0} then some ball around x contains no point of $\{x_1, x_2, ...\}$, which is a contradiction. It follows that $\{x_{n_0}\}$ is open (and closed) which contradicts the hypothesis. Note that if $f : X \to \mathbb{R}$ is continuous and $\int |f|\, d\mu < \infty$ then $\mu(\{x : |f(x)| > \alpha\}) < \infty$ for each $\alpha > 0$ which implies $\{x : |f(x)| > \alpha\}$ is empty for each $\alpha > 0$,so f is identically 0!.

Problem 340

If $f : \mathbb{R}^2 \to \mathbb{R}$ is increasing in each variable does it follow that f is Borel measurable?

No! Let $A = \{(x, -x) : x \in \mathbb{R}\}$ and B be a non-Borel subset of A. Let

$$f((x,y)) = \left\{ \begin{array}{c} 0 \text{ if } x < -y \\ 3 \text{ if } x > -y \\ 1 \text{ if } (x,y) \in B \\ 2 \text{ if } (x,y) \in A \backslash B \end{array} \right\}.$$

f is not measurable because $f^{-1}((0.5, 1.5)) = B$. It is easy to check that f is increasing in each variable.

Problem 341

Show that there exist continuous probability density functions $f_n, n = 1, 2, ...$ supported by $[0, 1]$ such that $\int_a^b f_n(x)dx \to b - a$ whenever $0 \le a \le b \le 1$ but $\int_A f_n(x)dx \nrightarrow m(A)$ for some Borel set A.

Let A be a Cantor-like set such that $0 < m(A) < 1$. We claim that Lebesgue measure m belongs to the closure of the (convex) set of all probability measures on $[0, 1]$ which have finite support in A^c. If this is false then there exists $f \in C[0, 1]$ such that $\int_0^1 f(x)dx < \int f d\mu$ for every μ which has finite support in A^c.. [This follows from a separation theorem applied to the dual of $C[0, 1]$ with the weak* topology]. In particular $\int_0^1 f(x)dx < f(y)$ for every $y \in A^c$. Since A has no interior A^c is dense. Hence $\int_0^1 f(x)dx \le f(y)$ for every y and strict inequality holds on a set of positive measure. This is a contradiction and so the claim is established. For each probability measure μ on $[0, 1]$ which have finite support in A^c we can find continous density functions g_n such that

$g_n dx \to \mu$ weakly. [For example if $x_0 \in A^c$ then δ_{x_0} is the weak limit of $\{g_n dx\}$ where $g_n(x) = \left\{ \begin{array}{c} n^2(x - x_0 + \frac{1}{n}) \text{ if } x_0 - \frac{1}{n} \leq x \leq x_0 \\ n^2(x_0 - x + \frac{1}{n}) \text{ if } x_0 \leq x \leq x_0 + \frac{1}{n} \\ 0 \text{ if } |x - x_0| < \frac{1}{n} \end{array} \right\}$. By taking convex combinations we get the same conclusion for any probability measure on $[0,1]$ which has finite support in A^c]. It follows that there exist continuous probability density functions $f_n, n = 1, 2, ...$ supported by $[0,1]$ such that $f_n(x)dx \to m$ and $f_n(x) = 0$ for all $x \in A$. Since $\int_A f_n(x)dx = 0$ for all n but $m(A) > 0$ we are done.

Problem 342

If X and Y are non-negative integer valued random variables such that $Et^{X+Y} = Et^X Et^Y$ for $0 \leq t \leq 1$ does it follow that X and Y are independent?

No!. Let $A_1, ..., A_9$ be the following sets: $A_1 = (0, 1/9), A_2 = (1/9, 1/6), A_3 = (1/6, 1/3), A_4 = (1/3, 1/2), A_5 = (1/2, 11/18), A_6 = (11/18, 2/3), A_7 = (2/3, 13/18)$,

$A_8 = (13/18, 8/9), (8/9, 1)$ and $X = Y = 1$ on $A_1, X = 1, Y = 2$ on $A_2, X = 1, Y = 3$ on $A_3, X = 2, Y = 1$ on A_4,

$X = Y = 2$ on $A_5, X = 2, Y = 3$ on $A_6, X = 3, Y = 1$ on $A_7, X = 3, Y = 2$ on $A_8, X = Y = 3$ on A_9. It is easily seen that both X and Y take the vaues $1, 2, 3$ with probabilities $1/3$ each. Hence their common moment generating function is given by $M(t) = \frac{t+t^2+t^3}{3}$. Also $Et^{X+Y} = (1/9)t^2 + (1/18 + 1/6)t^3 + (1/6 + 1/18 + 1/9)t^4 + (1/18 + 1/6)t^5 + (1/9)t^6$

$= (1/9)t^2 + (2/9)t^3 + (3/9)t^4 + (2/9)t^5 + (1/9)t^6 = (\frac{t+t^2+t^3}{3})^2 = Et^X Et^Y$.

Remark: it is trivial to find random variables X and Y which are not independent such that $Ee^{it(X+Y)} = Ee^{itX} Ee^{itY}$ $\forall t \in \mathbb{R}$ but X and Y are not independent: take $X = Y$ with characteristic function $e^{-|t|}$.

Problem 343

Suppose $X_n \geq 0, \{X_n\}$ is uniformly integrable and $X_n \to 0$ a.s. and in L^1. Can we conclude that $E(X_n|\mathcal{G}) \to 0$ a.s.?

No! Arrange the intervals $(\frac{i-1}{2^n}, \frac{i}{2^n}), 1 \leq i \leq 2^n, n \geq 1$ according to the dictionary order on the pairs (n, i). Call these sets $A_1, A_2, ...$ Let $B_n = (0, 1/n)$ and $X_n = nI_{A_n \times B_n}$ on $\Omega = (0,1) \times (0,1)$ with Borel sigma field and Lebesgue measure. . Then $X_n \geq 0, X_n \to 0$ at every point and $EX_n = m(A_n) \to 0$. Let $\mathcal{G}$ be the sigma field generated by the first projection which is $\{A \times (0,1) : A$ is Borel in $(0,1)\}$. A simple verification shows that $E(X_n|\mathcal{G}) = I_{A_n \times (0,1)}$. Of course $I_{A_n \times (0,1)}$ does not converge to 0 a.s. Note that $\{X_n\}$ is uniformly integrable but not dominated by any L^1 function. [If it is dominated then we would have $\limsup E(X_n|\mathcal{G}) \leq E(\limsup X_n|\mathcal{G})$. But the left side is 1 and the right side is 0!

Problem 344

If $X_n's$ are independent identically distributed positive random variables does it follow that $EX_1X_2... = (EX_1)(EX_2)....$ assuming that all the products and expectations exist.

No! Let $X_n's$ take values 1/2 and 3/2 with probabilty 1/2 each. Then $EX_n = 1/4 + 3/4 = 1$ so $(EX_1)(EX_2)... = 1$. We claim that $EX_1X_2... = 0$. Since $E(X_1X_2...X_{n+1}/X_1, X_2, ..., X_n) = X_1X_2...X_nEX_n = X_1X_2...X_n$ it follows that $\{X_1X_2...X_n\}$ is a non-negative martingale. Hence it converges a.s. We shall show that the limit is 0 a.s. thereby completing the proof. Let $N_n = \#\{k \leq n : X_k = 3/2\}$. Then $X_1X_2...X_n = (1/2)^{n-N_n}(3/2)^{N_n} = \frac{3^{N_n}}{2^n}$. By SLLN applied to $\{I_{X_k} = 3/2\}$ we see that $\frac{1}{n}N_n \to 1/2$. Hence $\log(\frac{3^{N_n}}{2^n}) = N_n \log 3 - n \log 2 = n(\frac{1}{n}N_n \log 3 - \log 2) \to\to \infty$ a.s. because $1/2 \log 3 - \log 2 < 0$.

Problem 345

Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and $f(x + y) - f(x)$ be continuous for all y in some set of positive measure. Show that f is continuous. Show that measurability cannot be dropped. If $f(x + y) - f(x)$ be continuous for all y in some set with a limit point can we conclude that f is continuous? Does there exist a set A of measure 0 such that if $f(x + y) - f(x)$ is continuous for all y in A then If f is necessarily continuous? If A is at most countable show that there exist f such that $f(x + y) - f(x)$ is continous for all x in A but f is not continuous. If $f : \mathbb{R} \to \mathbb{C}$ is continuous and $f(x+y)-f(x)$ is an entire function for all y in some set with a limit point can we conclude that f is entire?

We first note that $\{y : f(x+y)-f(x)$ is continuous$\}$ is an additive subgroup of $\mathbb{R}$. Hence, if it contains a set of positive measure than it must be the whole of $\mathbb{R}$. [Because $m(A) > 0$ implies $A - A$ contains an interval around 0]. Now let $g(x) = e^{-|f(x)|}$. The function $t \to \int\limits_0^1 |g(x - t) - g(x)| \, dx$ is continuous.[See Rudin's Real and Complex Analysis, for example]. Let $x_n \to 0$. Then $\int_0^1 |g(x + x_n) - g(x)| \, dx \to 0$. We claim that $|f(x_n)| \nrightarrow \infty$. If $|f(x_n)| \to \infty$ then for any x $|f(x + x_n)| \to \infty$ too (because $f(x_n + x) - f(x_n) \to f(x) - f(0)$ by hypothesis, so $f(x_n + x) - f(x_n)\}$ is bounded). It follows that $g(x + x_n) \to 0$ and Dominated Convergence Theorem gives $\int_0^1 |g(x)| \, dx = 0$. This is obviously a contradiction. Hence $x_n \to 0$ implies $\{f(x_n\}$ is bounded. In other words, f is bounded in a neighbourhood of 0. Now $f(x + y) - f(x)$ is bounded in a neighbourhood of 0 and hence $f(x + y)$ is bounded in a neighbourhood of 0. This is true for each y and it follows easily that f is bounded on compact sets.

Now $\int\limits_0^1 [f(x+y)-f(x)]dy$ is continuous by Dominated Convergence Theorem. This means $\int\limits_x^{x+1} f(y)dy - f(x)$ is continuous. Since the first term is continuous it follows that f is continuous. The conclusion may fail when f is not measurable: there exists additive non-measurable (hence non-continuous) functions. If $f(x+y)-f(x)$ be continuous for all y in some set with a limit point can we cannot conclude that f is continuous: let $f = I_{\mathbb{Q}}$ so that $f(x+y)-f(x) = 0$ for every rational number y. If f is continuous and $f(x+y)-f(x)$ is an entire function for all y in some set with a limit point can we cannot conclude that f is entire: let $f(z) = \bar{z}$. If A is atmost countable then the subgroup B of $(\mathbb{R},+)$ generated by A is also at most countable. Let $f = I_B$. Then $f(x+y)-f(x)$ is the zero function for all $x \in A$ but f is not continuous since $B \neq \mathbb{R}$. If A is the Cantor set of measure 0 then the group generated by A is $\mathbb{R}$ (becuase $A+A=[0,2]$) hence the continuity of $f(x+y)-f(x)$ for y in A implies the same for all y which implies continuity of f. Hence there does exist a set A of measure 0 such that if $f(x+y)-f(x)$ is continuous for all y in A then f is necessarily continuous.

Problem 346

Let X and Y be random variables such that $X, Y, X+Y, X-Y$ all have the same distribution. If the common distribution has finite mean show that $X=0$ a.s. Prove that the assumption on finiteness of the mean cannot be dropped.

Since $2E|X| = E|(X+Y)+(X-Y)| = E|(X+Y)| + E|(X-Y)|$ it follows that $X+Y = Z(X-Y)$ for some non-negative random variable Z. Hence

$X(Z-1) = Y(1+Z)$. Noting that $|Z-1| \leq 1+Z$ we get $|Y|(1+Z) \leq |X|(1+Z)$ which implies $|Y| \leq |X|$. Since both sides have the same mean we get $|Y| = |X|$. This implies $|X|(1+Z) = |X||1-Z|$ so $1+Z = |1-Z|$ when $X \neq 0$. In other words $X \neq 0$ implies $Z = 0$ and $X(0-1) = Y(1+0)$ or $Y = -X$. But $X = 0$ implies

$Y(1+Z) = 0$ so $Y = 0$. Hence $Y = -X$ in both cases and $X+Y = 0$ a.s. It follows that $X = 0$ a.s.

For the counterexample let U, V be i.i.d. with characteristic function $e^{-|t|}$. Let $X = \frac{U+V}{2}$ and $Y = \frac{U-V}{2}$.

Problem 347

Let (X,d) be a separable metric space, A be a closed subset such that every subset of A is open in A. Show that A is at most countable.

If $B \subset A$ then I_B is continuous on A. By Tietze Theorem we can extend it to a real continuous function f_B on X. The map $B \to \{f_B(x_n)\}$, where $\{x_n\}$

is a countable dense subset of X, is injective. There are c elements in $\mathbb{R}^{\mathbb{N}}$ and so there are at most c subsets of A. Hence A is at most countable. Alternate proof: A is itself a separable metric space with discrete topology. The open cover formed by sigletons has a countable subcover.

Problem 348

Let $\{a_n\} \subset \mathbb{R}$ with $\sum_n |a_n| < \infty$. Let $S = \{\sum_{n \in I} a_n : \ddot{I} \subset \mathbb{N}\}$. Is S closed? Is it necessarily a closed interval if $a_n > 0$ for all n?.

Yes. The map θ which takes $\{\delta_n\} \in \{0,1\}^{\mathbb{N}}$ to $\sum_{n \in I} a_n$ where $I = \{n : \delta_n = 1\}$ is continuous on $\{0,1\}^{\mathbb{N}}$ with the product topology: given $\epsilon > 0$ choose N such that $\sum_{n>N} |a_n| < \epsilon$. If $\{\delta_n^{(j)}\} \to \{\delta_n\}$ in $\{0,1\}^{\mathbb{N}}$ then there exists j_0 such that $\delta_n^{(j)} = \delta_n$ for $1 \leq n \leq N$ and $j \geq j_0$. It follows that $\left|\theta(\{\delta_n^{(j)}\}) - \theta(\{\delta_n\})\right| \leq 2\epsilon$ for $j \geq j_0$. Tychonoff's Theorem shows that the range S of this map is compact. [Note that S is actually a perfect set]. If $a_n = \frac{2}{3^n}$ then S is the Cantor set, so

$j \geq j_0$, so S need not be an interval. Z. Nitecki has an article which gives a complete characterization of subsums of series whose $n-th$ term tends to 0.

Problem 349

Let X be a Banach space, M closed subspace of X^* such that $x^*(x) = 0$ for all $x^* \in M$ implies $x = 0$. Prove that the following are equivalent:

a) there exists $c \in (0, \infty)$ such that $c\|x\| \leq \sup\{|x^*(x)| : x^* \in M, \|x^*\| = 1\}$ for all $x \in X$

b) $\{x^{**} \in X^{**} : x^{**}(x^*) = 0 \text{ for all } x^* \in M\} + X$ is a closed subspace of X^{**}.

We first observe that $\{x^{**} \in X^{**} : x^{**}(x^*) = 0 \text{ for all } x^* \in M\} \cap X = \{0\}$ by hypothesis. Hence the sum in b) is a direct sum. Suppose a) holds. Define a new norm on $\{x^{**} \in X^{**} : x^{**}(x^*) = 0 \text{ for all } x^* \in M\} + X$ by $\|x^{**} + x\|_1 = \|x^{**}\| + \|x\|$. It is easy to see that this is a complete norm. If we show that the new norm is equivalent to the original norm we can conclude that $\{x^{**} \in X^{**} : x^{**}(x^*) = 0 \text{ for all } x^* \in M\} + X$ is complete, hence closed in X^{**}.

Of course $\|x^{**} + x\|_1 \geq \|x^{**} + x\|$. We claim that $\|y^{**}|M\| = d(y^{**}, N)$ for all $y^{**} \in X^{**}$ where $N = \{x^{**} \in X^{**} : x^{**}(x^*) = 0 \text{ for all } x^* \in M\}$. To see this we note that if $x_0^{**} \in N$ then $|y^{**}(x^*)| = |y^{**}(x^*) - x_0^{**}(x^*)| \leq \|y^{**} - x_0^{**}\| \|x^*\|$ for $x^* \in M$ so $\|y^{**}|M\| \leq \|y^{**} - x_0^{**}\|$ for all $x_0^{**} \in N$ which implies $\|y^{**}|M\| \leq d(y^{**}, N)$. On the other hand there exists $z^{**} \in X^{**}$ such that $z^{**} = y^{**}$ on M and $\|z^{**}\| = \|y^{**}|M\|$. We have $d(y^{**}, N) \leq \|y^{**} - (y^{**} - z^{**})\| = \|z^{**}\| = \|y^{**}|M\|$. We have proved the claim. For $x \in X$ and $x^{**} \in N$ we have $\|x^{**} + x\| \geq d(x, N) = \|x|M\|$ (by the claim, with $y^{**} = x$) which gives $\|x^{**}+x\| \geq \sup\{|x^*(x)| : x^* \in M, \|x^*\| = 1\} \geq c\|x\|$ by a). This gives $\|x^{**}+x\|_1 = \|x^{**}\| + \|x\| \leq \|x^{**}+x\| + 2\|x\| \leq (1+\frac{2}{c})\|x^{**}+x\|$. We have proved the equivalence of the two norms on $N+X$. This proves a) implies b). Now suppose b) holds. Since $X+N$ is complete and $\|x^{**}+x\|_1 \geq \|x^{**}+x\|$ it follows that the two norms are equivalent (by Open Mapping Theorem) and hence there exists $C \in (0,\infty)$ such that $\|x^{**}\| + \|x\| = \|x^{**}+x\|_1 \leq C\|x^{**}+x\|$. In particular $\|x\| \leq C\|x^{**}+x\|$ for all $x^{**} \in N$. Hence $\|x\| \leq Cd(x,N) = C\|x|M\| = C\sup\{|x^*(x)| : x^* \in M, \|x^*\| = 1\}$ and a) holds with $c = \frac{1}{C}$.

Problem 350

Suppose $\{x_n\}$ is a sequence of unit vectors in a Hilbert space H such that $\liminf \|x_n + x\| \geq \|x\|$ for all $x \in H$. Show that $x_n \to 0$ weakly. Is this true in Banach spaces?

We have $\liminf[1 + 2k\operatorname{Re} < x_n, x >] \geq 0$ for every x and every positive integer k. This gives $\operatorname{Re} < x_n, x > \to 0$. [If $\operatorname{Re} < x_{n_j}, x > \geq \delta > 0$ for some $\{n_j\}$ replace x by $-x$ to get a contradiction. If $-\operatorname{Re} < x_{n_j}, x > \geq \delta > 0$ then also we have a contradiction]. In the real case change x to $-x$ and in the complex case change x to ix to see that $< x_{n_j}, x > \to 0$.

The conclusion may fail in a Banach space: let $X = L^1[0,1], f_n = nI_{(0,\frac{1}{n})}$. Then $\int |f_n + g| = \int_0^{1/n} |n+g| + \int_{1/n}^1 |g|$

$= \int_0^{1/n} [|n+g| - |g|] + \int_0^1 |g|$ and $\int_0^{1/n} [|n+g| - |g|] \geq \int_0^{1/n} [n - 2|g|] = 1 - 2\int_0^{1/n} |g| \to 1$ so the hypothesis is satisfied. However the constant function 1 is in X^* and $\int (f_n)(1) = 1$ for all n so $f_n \nrightarrow 0$ weakly. [If we redefine $f_n's$ by $f_n(x) = 2n(\frac{1}{n} - x)$ for $0 \leq x \leq \frac{1}{n}$ and 0 elsewhere we get a similar conclusion

in $C[0,1]$].

Problem 351

Let X be the Banach space of all bounded continuous functions from $\mathbb{R}$ into itself. Show that there is a linear map $\Lambda : X \to \mathbb{R}$ such that $\Lambda f \geq 0$ for any non-negative function f in X but the equation $\Lambda f = \int f d\mu$ $(f \in X)$ does not hold for any measure μ.

If such a measure exists it is necessarily a finite positive measure. Let $x_n = n + \frac{1}{2}, n \geq 1$. Let $p(f) = \limsup f(x_n)$ for any $f \in X$. Note that

$p(f+g) \leq p(f)+p(g)$ and $p(cf) = cp(f)$ for $c \geq 0$. The map Λ which takes a constant function to the constant value is a linear map on the one-dimensional subspace of constants satisfying the condition $\Lambda f \leq p(f)$. By Hahn Banach Theorem there exists a linear map Λ on X mapping 1 to 1 such that $\Lambda(f) \leq p(f) = \limsup f(x_n)$ for any $f \in X$. Changing f to $-f$ we get $-\Lambda(f) \leq \limsup\{-f(x_n)\} = -\liminf f(x_n)$. Hence $\liminf f(x_n) \leq \Lambda(f) \leq \limsup f(x_n)$. In particular Λ is a positive linear functional. [Also $|\Lambda f| \leq \|f\|_\infty$]. Suppose $\Lambda f = \int f d\mu$ $(f \in X)$. Let f_n be a continuous function $: \mathbb{R} \to [0,1]$ such that $f_n(x) = 1$ for $|x| \leq n$ and $f_n(x) = 0$ for $|x| > n+1$. Note that $p(f_n) = \limsup_{m\to\infty} f_n(x_m) = 0$ for each n. Also $f_n \to 1$ pointwise. By Bounded Convergence Theorem we most have $\Lambda f_n = \int f_n d\mu \to \int 1 d\mu = \Lambda 1 = 1$. However $\Lambda f_n \leq p(f_n) = 0$ for each n.

Problem 352

Show that there exists non-zero elements f and g in $L^1(\mathbb{R})$ such that $f*g = 0$. However, $f*f = 0$ implies $f = 0$.

The second part follows by taking Fourier transform. Let $f(x) = \frac{1-\cos x}{x^2}$ for $x \neq 0$ and $f(0) = 0$. Let $g(x) = e^{2ix}f(x)$. Then the Fourier transform of $f*g$ is 0.

Problem 353

Let $f : \mathbb{R} \to \mathbb{R}$ be a given map and let τ be the smallest topology on $\mathbb{R}$ which makes f continuous. Suppose $g : (\mathbb{R}, \tau) \to \mathbb{R}$ is continuous. Show that there exists a unique continuous function $h : f(\mathbb{R}) \to \mathbb{R}$ such that $g = h \circ f$.

Suppose $f(x) = f(y)$. If $g(x) \neq g(y)$ then there exist disjoint open sets U, V in $\mathbb{R}$ such that $g(x) \in U$ and $g(y) \in V$. By hypothesis there exist open sets U_1, V_1 in $\mathbb{R}$ such that $g^{-1}(U) = f^{-1}(U_1)$ and $g^{-1}(V) = f^{-1}(V_1)$. Then $f(x) \in U_1$ and $f(y) \in V_1$. Since $f(y) \in V_1$ we have $f(x) \in V_1$ too and $x \in f^{-1}(V_1) = g^{-1}(V)$. Thus $g(x) \in V$ contradicting the fact that $g(x) \in U$ and $U \cap V = \emptyset$. This shows that $f(x) = f(y)$ implies $g(x) = g(y)$. Hence there exists a unique function $h : f(\mathbb{R}) \to \mathbb{R}$ such that $g = h \circ f$. We now prove that h is continuous. Suppose $f(x_n) \to f(x)$. If S is an open set containing $g(x)$ then there exists an open set T such that $g^{-1}(S) = f^{-1}(T)$. Since $x \in g^{-1}(S)$ we see that $f(x) \in T$. Hence $f(x_n) \in T$ for all n sufficiently large. But then $x_n \in f^{-1}(T) = g^{-1}(S)$ and $g(x_n) \in S$ for all n sufficiently large. Thus $g(x_n) \to g(x)$. This proves that h is continuous on $f(\mathbb{R})$.

Remark: the function h may not extend continuously to $\mathbb{R}$. For example of $f(x) = e^{-|x|}$ and $g(x) = e^{|x|}$ the the hypothesis is satisfied and $h(t) = \frac{1}{t}$ for all $t \in (0,1] \equiv f(\mathbb{R})$. Compare this situation with the following: if Ω is a non-empty set and $f : \Omega \to \mathbb{R}$ is a given map then for any measurable function $g : (\Omega, \mathcal{G})$ where $\mathcal{G}$ is the sigma field generated by f there exists a measurable function $h : \mathbb{R} \to \mathbb{R}$ such that $g = h \circ f$. [This is proved easily by a simple function approximation].

Problem 354 [World turned upside down!]

a) Let $p > 1$ and $q = 1 - p$. Let g be a positive measurable function with $\int g > 0$. Let f be a non-negative measurable function which is integrable. Show that $\int f^p g^q \geq (\int f)^p (\int g)^q$.

b) Let $0 < p < 1$, f and g non-negative measurable functions such that $f + g > 0$ and $\int f^p + \int g^p < \infty$. Show that $(\int (f+g)^p)^{1/p} \geq (\int f^p)^{1/p} + (\int g^p)^{1/p}$. The integrals are w.r.t. any positive measure.

We have $\int f = \int [f g^{q/p}][g^{-q/p}] \leq [\int [f g^{q/p}]^p]^{1/p} \int (g^{-q/p})^{p/(p-1)}]^{1-1/p} = [\int f^p g^q]^{1/p} [\int g]^{1-1/p}$. This gives $\int f^p g^q \geq (\int f)^p (\int g)^q$ proving a). Proof of b): we have $\int (f+g)^p = \int f(f+g)^{p-1} + \int g(f+g)^{p-1} = \int (f^p)^{1/p} \{(f+g)^p\}^{1-1/p} + \int (f^p)^{1/p} \{(f+g)^p\}^{1-1/p}$

$\geq (\int f^p)^{1/p} (\int (f+g)^p)^{1-1/p} + (\int g^p)^{1/p} (\int (f+g)^p)^{1-1/p}$ by a) with p changed to $1/p$. Hence $(\int (f+g)^p)^{1/p} \geq (\int f^p)^{1/p} + (\int g^p)^{1/p}$.

Problem 355

Let μ be a complex measure on $(\mathbb{R}, \mathcal{B})$. Let μ_c be the measure $E \to \operatorname{Re}\{c\mu(E)\}$ for each $c \in \Delta \equiv \{z \in \mathbb{C} : |z| \leq 1\}$. Show that the supremum of the family of real measures $\{\mu_c : c \in \Delta\}$ is $|\mu|$.

Remark: setwise supremum of a family of measures is not a mesure in general. What we are asked to show is the following: $\mu_c(E) \leq |\mu|(E)$ for each E and if ν is a real mesure such that $\mu_c(E) \leq \nu(E)$ for each E then $|\mu|(E) \leq \nu(E)$ for each E.

It is obviuos that $\mu_c(E) \leq |\mu|(E)$ for each E. If ν is as above let $\lambda = |\mu| + |\nu|$. Let $g = \frac{d\mu}{d\lambda}, h = \frac{d\nu}{d\lambda}$. Then $\int_E \operatorname{Re}(cg) d\lambda = \operatorname{Re} c \int_E g d\lambda = \mu_c(E) \leq \mu(E) = \int_E h d\lambda$. Since this holds for each Borel set E we get $|\operatorname{Re}(cg)| \leq h$ a.e. $[\lambda]$. Since Δ is separable this gives $|g| \leq h$ a.e. $[\lambda]$. This implies that μ is necessarily a positive measure and $|\mu| \leq \nu$.

Problem 356
See also problems 589-592

On $[0,1]\times[0,1]$ we cannot write $|x-y|$ as $\sum_{i=1}^{n} f_i(x)g_i(y)$ with $n \in \mathbb{N}$, $f_i's$ and $g_i's$ continuous

- We first show that the functions $|x-a_i|, 1 \leq i \leq k$ are linearly independent if the numbers $a_1, a_2, ..., a_k \in [0,1]$ are distinct. This is easily seen from the fact that $|x-a_i|$ cannot be written as a linear combination of $|x-a_j|, j \neq i$ because $|x-a_i|$ is not differentiable at a_i whereas $|x-a_j|$ is differentiable at a_i for $j \neq i$. Now suppose $|x-y| = \sum_{i=1}^{n} f_i(x)g_i(y)$. Choose $n+1$ distinct points $a_1, a_2, ..., a_{n+1}$. Let $\alpha_{i,j} = g_i(a_j), 1 \leq i \leq n, 1 \leq j \leq n+1$. The system of n equations $\sum_{j=1}^{n+1} \beta_j \alpha_{i,j} = 0, 1 \leq i \leq n$ in $n+1$ variables $\beta_1, \beta_2, ..., \beta_{n+1}$ has a non trivial solution. When $\beta_j's$ satisfy these equations we have $0 = \sum_{j=1}^{n+1}\sum_{i=1}^{n} \beta_j f_i(x) g_i(a_j) = \sum_{j=1}^{n+1} \beta_j |x-a_j|$ contradicting the fact that $|x-a_j|, 1 \leq j \leq n+1$ are linearly independent.

Problem 357

See also problems 589-592

If $X, X_n, n = 1, 2, ...$ are random variables taking values in $[0,1]$ such that $E|X_n - a| \to E|X-a|$ for each $a \in [0,1]$ show that $X_n \xrightarrow{w} X$. [$\xrightarrow{w}$ denotes weak convergence]

Claim: if $f : [0,1] \to \mathbb{R}$ is a piecewise linear continuous function then there exist points $0 = a_0 < a_1 < ... < a_N = 1$ and real numbers $c_0, c_1, ..., c_N$ such that $f(x) = \sum_{i=1}^{N} c_i |x-a_i| + c_0$. Granting this for the moment we get $Ef(X_n) \to Ef(X)$ for any piece-wise linear continuous function. Since any continuous function from $[0,1]$ to $\mathbb{R}$ can be uniformly approximated by piece-wise linear continuous functions it follows that $Ef(X_n) \to Ef(X)$ for any continuous function f which proves wek convergence. To prove the claim let f be linear on $[a_{i-1}, a_i]$ for $1 \leq i \leq N$ where $0 = a_0 < a_1 < ... < a_N = 1$. Since

$\sum_{i=1}^{N} c_i |x-a_i| + c_0$ is also linear on $[a_{i-1}, a_i]$ for $1 \leq i \leq N$ (for any choice of $c_i's$) it suffices to show that $f(0) = \sum_{i=1}^{N} c_i a_i + c_0$ and the slope $\sum_{i=1}^{j-1} c_i - \sum_{i=j}^{N} c_i$ of $\sum_{i=1}^{N} c_i |x-a_i| + c_0$ on $[a_{i-1}, a_i]$ coincides with the slope, say m_j of f on that interval. We define $c_j = \frac{m_{j+1}-m_j}{2}$ for $1 \leq j \leq N-1$ and $c_N = -\frac{m_1+m_N}{2}$. Finally we choose c_0 such that $f(0) = \sum_{i=1}^{N} c_i a_i + c_0$. This completes the proof.

Problem 358

Consider the sequence $\{f_n\}$ in $\{0,1\}^{\mathbb{R}}$ defined by $f_n(x) = [2^n x] - 2[2^{n-1}x]$. Prove that $\{f_n\}$ has no subsequence converging pointwise to any function.

Suppose $f_{n_k} \to f$ pointwise on $\mathbb{R}$. Of course each f_n is measurable and hence f is measurable. If $\frac{i-1}{2^{n-1}} \leq x < \frac{i}{2^{n-1}}$ then either $\frac{2i-2}{2^n} \leq x < \frac{2i-1}{2^n}$ or $\frac{2i-1}{2^n} \leq x < \frac{2i}{2^n}$. In the first case $f_n(x) = 0$ and in the second case $f_n(x) = 1$. Let $\frac{2l-2}{2^{n+m}} \leq x + \frac{1}{2^m} < \frac{2l-1}{2^{n+m}}$. Then $f_{n+m}(x + \frac{1}{2^m}) = (2l-2) - 2(l-1) = 0$ and since $\frac{2l-2-2^n}{2^{n+m}} \leq x < \frac{2l-1-2^n}{2^{n+m}}$ we have $\frac{2i-2}{2^n} \leq x < \frac{2i-1}{2^n}$ where $i = l - 2^{n-1}$ we get $f_n(x) = 0$ too. Similarly if $\frac{2l-1}{2^{n+m}} \leq x + \frac{1}{2^m} < \frac{2l}{2^{n+m}}$ then $f_{n+m}(x + \frac{1}{2^m}) = 1 = f_{n+m}(x)$. Hence $f_{n+m}(x + \frac{1}{2^m}) = f_{n+m}(x)$ for all $x \in \mathbb{R}$ for all $m \in \mathbb{N}$. It follows that $f(x+d) = f(x)\ \forall x \in \mathbb{R}, \forall d \in D$ where D is the set of all dyadic rationals. Thus f is a function with values in $\{0,1\}$ which has every dyadic rational as a period. We prove that such a function cannot be measurable. Let $A = \{x : f(x) = 1\}$ and $B = \{x : f(x) = 0\}$. Then $\int |I_{d+A} - I_A| = 0$ if $d \in D$. By the continuity of translates in L^1 we conclude that $\int |I_{y+A} - I_A| = 0$ for all $y \in \mathbb{R}$. We claim that I_A is constant almost everywhere. Let $\phi_n(t) = \frac{1}{\sqrt{2\pi n}} e^{-t^2/2n}$. Then it is easy to see that $\phi_n * I_A$ is a continuous function which has every real number as a period. It follows that $\phi_n * I_A$ is a constant for each n. Also $\phi_n * I_A \to I_A$ in L^1 so I_A is a constant. To arrive at a contradiction from this we prove that $m((0,1) \cap f_n^{-1}\{0\}) = m((0,1) \cap f_n^{-1}\{1\}) = \frac{1}{2}$. Indeed for $0 < x < 1$ we have $f_n(x) = 0$ iff $x \in \frac{2i-2}{2^n} \leq x < \frac{2i-1}{2^n}$ for some i. Hence $m((0,1) \cap f_n^{-1}\{0\}) = \sum_{i \text{ even }, 1 \leq i \leq 2^n} \frac{1}{2^n} = \frac{1}{2}$ which of course implies $m((0,1) \cap f_n^{-1}\{1\}) = \frac{1}{2}$. Now the fact that $f_{n_k} \to f$ pointwise implies $m(\{f > 1/2\}) \leq \liminf m(\{f_{n_k} > 1/2\}) = \frac{1}{2}$ and $m(\{f < 1/2\}) \leq \liminf m(\{f_{n_k} < 1/2\}) = \frac{1}{2}$ so f cannot be a.e. constant.

Remark: by Tychonoff's Theorem there is a *subnet* of $\{f_n\}$ which converges. The limiting fucntion is non-measurable by above argument.

Problem 359

Give an example of a map ϕ from $[0,1]$ into a (necessarily non-separable) Hilbert space H with the following properties:

a) ϕ is not Lebesgue measurable

b) for each $x \in H$ the map $t \to < x, \phi(t) >$ is Lebesgue measurable and, in fact, it is 0 a.e.

c) $\phi^{-1}(B)$ is a Borel set for each open ball B in H.

Let H be a real Hilbert space with an orthonormal basis $\{e_t\}_{0 \leq t \leq 1}$ indexed by $[0,1]$. Let $\phi(t) = e_t$. Let A be a non-measurable set in $[0,1]$. We claim that $x \to \sum_{t \in A} < x, e_t >^2$ is continuous on H. To see this we just have to apply triangle inequality: $\left| \sqrt{\sum_{t \in A} < x, e_t >^2} - \sqrt{\sum_{t \in A} < y, e_t >^2} \right| \leq \sqrt{\sum_{t \in A} < x - y, e_t >^2} \leq \|x - y\|$. Now $\{x : \sum_{t \in A} < x, e_t >^2 \neq 0\}$ is an open set in H whose inverse image under ϕ is A. This proves a). If $x \in H$ the map $t \to < x, \phi(t) >$ is 0 a.e. because it is 0 except on a countable set. Finally $\{t : \|e_t - x\| < r\} = \{t : 1 + \|x\|^2 - 2 < x, e_t >< r^2\}$ is either a subset of $\{t : < x, e_t > \neq 0\}$ or the complement of such a set.

Problem 360

Let $f : \mathbb{R} \to \mathbb{R}$. The following two statements are equivalent:

a) there exists $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ a.e. and g is continuous a.e.

b) there exists a set A of measure 0 such that the restriction of f to A^c is continuous [w.r.t. the topology on A^c induced by the usual topology on $\mathbb{R}$].

To see that a) implies b) just take $A = \{x : f(x) \neq g(x)\}$. For the converse we define g by $g(t) = \liminf_{y \in A^c, y \to t} f(y)$. [Note that A^c is dense, so there exist sequences $\{y_n\} \subset A^c$ converging to t]. It is clear that $g = f$ on A^c. Hence $f = g$ a.e.. We now prove that g is continuous at each point of A^c. Let $t \in A^c$ and $\epsilon > 0$. There exists $\delta > 0$ such that $|f(y) - f(t)| < \epsilon$ if $y \in A^c$ and $|y - t| \leq \delta$. Let $s \in \mathbb{R}$ and $|s - t| < \delta/2$. We claim that $|g(s) - g(t)| \leq \epsilon$. We have $f(y) < f(t) + \epsilon = g(t) + \epsilon$ whenever $y \in A^c$ and $|y - t| \leq \delta$. The same inequality holds if $y \in A^c$ and $|y - s| \leq \delta/2$. Hence $g(s) = \liminf_{y \in A^c, y \to s} f(y) \leq g(t) + \epsilon$. Also $f(y) > f(t) - \epsilon = g(t) - \epsilon$ $y \in A^c$ and $|y - t| \leq \delta$, hence whenever $y \in A^c$ and $|y - s| \leq \delta/2$. Therefore $g(s) \geq f(t) - \epsilon$. This completes the proof.

Remark: above conditiions imply that f is Lebesgue measurable: approximate g by step functions. However not every Lebesgue measurable function satisfies a) and b).

Problem 361

Let x and y be unit vectors in a normed linear space X. Let $p \in [1, \infty)$ and $0 \leq t \leq 1$. Show that $\|x - t^p y\| \leq 3p \|x - ty\|$ and $\|x - ty\|^p \leq 2^p \|x - t^p y\|$.

Writing t^p as $\alpha t + (1-\alpha)0$ with $\alpha = t^{p-1}$ we get $\|x - t^p y\| \leq t^{p-1} \|x - ty\| + (1 - t^{p-1})$. Now $(1 - t^{p-1}) = (1-t)(p-1)u^{p-2}$ for some $u \in [t, 1]$ and $(1-t)(p-1)u^{p-2} \leq (1-t)(p-1)t^{p-2}$. If $p \geq 2$ then $(1-t)(p-1)u^{p-2} \leq (1-t)(p-1) \leq \|x - ty\| (p-1)$ and $\|x - t^p y\| \leq t^{p-1} \|x - ty\| + \|x - ty\| (p-1) \leq p \|x - ty\|$. If $1 \leq p < 2$ we divide the proof into two cases: if $t \geq 1/3$ then $(1 - t^{p-1}) = (1-t)(p-1)u^{p-2} \leq (1-t)(p-1)3^{2-p} \leq 3(1-t)(p-1)$ so $\|x - t^p y\| \leq t^{p-1} \|x - ty\| + 3(1-t)(p-1)$
$\leq t^{p-1} \|x - ty\| + 3(p-1) \|x - ty\| \leq 3p \|x - ty\|$. If $t < 1/3$ then $\|x - t^p y\| \leq 1 + t^p \leq 2 \leq 2p \leq 3p(1-t) \leq 3p \|x - ty\|$. For the second part we have $\|x - ty\|^p = \|x - ty\|^{p-1} \|x - ty\| \leq (1+t)^{p-1}[\|x - t^p y\| + (t - t^p)] \leq 2^p \|x - t^p y\|$ because $[2^p - (1+t)^{p-1}] \|x - t^p y\| \geq [2^p - (1+t)^{p-1}][1 - t^p] \geq (1+t)^{p-1}(1 - t^p) \geq (1+t)^{p-1}(t - t^p)$.

Problem 362

Give a proof of the spectral radius formula for metrices without using Banach Algebra Theory. You may use the fact that an analytic function on $\{z : |z| < R\}$ has a power series expansion on that disc.

Let A be an $N \times N$ complex matrix and $\rho = \sup\{|\lambda| : \lambda$ is an eigen value of $A\}$. Spectral radius formula says $\rho = \lim_{n \to \infty} \|A^n\|^{1/n}$. If λ is an eigen value and $Ax = \lambda x, x \neq 0$ then $|\lambda|^n \|x\| = \|A^n x\| \leq \|A^n\| \|x\|$ so $|\lambda| \leq \|A^n\|^{1/n}$ for every n. It follows that $\rho \leq \|A^n\|^{1/n}$ for every n. Now $0 < |\lambda| < 1/\rho$ implies λ^{-1} is not an eigen value (by definition of ρ) and so $(I - \lambda A)^{-1}$ exists. Also $(I - \lambda A)^{-1} = \frac{1}{\det(I - \lambda A)} adj(I - \lambda A)$. It follows from this that $(I - \lambda A)^{-1}$ is analytic in $\{z : |z| < \rho\}$. For $|\lambda|$ sufficiently small we have $(I - \lambda A)^{-1} = \sum_{n=0}^{\infty} \lambda^n A^n$. This can be seen by noting that the series converges for $|\lambda| < \|A\|^{-1}$ and we have the identity $(I - \lambda A) \sum_{n=0}^{\infty} \lambda^n A^n = I$. It follows that the formula $(I - \lambda A)^{-1} = \sum_{n=0}^{\infty} \lambda^n A^n$ is valid for $|\lambda| < 1/\rho$. The radius of convergence of $\sum_{n=0}^{\infty} \lambda^n a_{ij}^{(n)}$, where $a_{ij}^{(n)}$ is the (i,j) element of A^n, is $[\limsup \left|a_{ij}^{(n)}\right|^{1/n}]^{-1}$. Since this radius is at least ρ^{-1} we get $\rho \geq \limsup \left|a_{ij}^{(n)}\right|^{1/n}$. Taking maximum over (i,j) we get $\rho \geq \limsup \|A^n\|^{1/n}$ where we have taken the definition of $\|A\|$ as

$\max\{|a_{ij}| : 1 \leq i, j \leq N\}$. Any other norm on $N \times N$ matrices will yield the same formula since all norms on finite dimensional spaces are equivalent.

Remark: we don't really need power series reprsentations of analytic functions. We used only the following fact which is elementary: let p be a polynomial, $c_i \in \mathbb{C}\backslash\{0\}(1 \leq i \leq N)$ and $f(z) = \frac{p(z)}{(z-c_1)(z-c_2)...(z-c_N)}$. Then f has a power series expansion for $|z| < \min\{|c_i| : 1 \leq i \leq N\}$. In particular, Cauchy's Theorem and its consequences have been avoided completely in this proof.

Problem 363

If Y is dense in a Hausdorff space X and if Y is locally compact in the relative topology from X show that Y is open in$X$. Hence show that a locally compact subgroup of a Hausdorff topological group is closed.

Let $y \in Y$. There is an open set U in X such that $y \in U$ and the $Y-$ closure Z of $U \cap Y$ is compact. Note that $Z \subset Y, Z$ is closed in X and $U \cap Y \subset Z$. We claim that $\bar{U} = [U \cap Y]^{-}$ (where $\bar{A}$ is the closure of A). If $u \in \bar{U}$ and V is an open set containing u then $V \cap U \cap Y$ is non-empty because $V \cap U$ is a non-empty open set and Y is dense in X. This proves the claim. Now $U \subset \bar{U} = [U \cap Y]^{-} \subset \bar{Z} = Z \subset Y$. Thus y is an interior point of Y. For the second part let H be a locally compact subgroup of a Hausdorff topological group G. Then $\bar{H}$ is a Hausdorff topological group and H is dense in $\bar{H}$. By the first part we conclude that H is open in $\bar{H}$. But an open subgroup is always closed so H is closed in $\bar{H}$, hence in G. [If H_0 is an open subgroup of G then $H_0^c = \bigcup\limits_{x \notin H_0} xH_0$ and this union is open. Hence H_0 is closed]

Problem 364

Let A be a closed subgroup of S^1 under multiplication. Show that $A = S^1$ or else A is a finite set.

Let $B = \{x \in \mathbb{R} : e^{2\pi ix} \in A\}$. Then B is a subgroup of $(\mathbb{R}, +)$. If B is dense in $\mathbb{R}$ then A is dense in S^1 because the map $p : \mathbb{R} \to S^1$ defined by $p(x) = e^{2\pi ix}$ is continuous. In that case $A = S^1$. If B is not dense then there exists $a > 0$ such that $B = \{na : n \in \mathbb{Z}\}$. But then $A = \{c^n : n \in \mathbb{Z}\}$ where $c = e^{2\pi ia}$. If a is irrational then $\{c^n : n \in \mathbb{Z}\}$ is dense and $A = S^1$. If a is rational then A is a finite set. [The fact that $\{c^n : n \in \mathbb{Z}\}$ is dense when a is irrational is a standard. Any book on Ergodic Theory contains a proof].

Problem 365

Let H be a closed subgroup of $(\mathbb{R}^n, +)$. If $H \cap L$ is a discrete subspace of L for every line L through the origin show that H is discrete.

Remark: if H be a closed subgroup of $(\mathbb{R}^n, +)$ which is not discrete then $H \cap L$ is a not discrete subspace of L for some line L through the origin. This implies that $H \cap L$ is dense in L. Since H is closed we get $L \subset H$. Thus H contains an entire line through the origin.

We assume that H is not discrete and prove that $H \cap L = L$ for at least one line L through the origin. If $\{0\}$ is an isolated point of H so is every other singleton subset of H and H is discrete. Hence $\{0\}$ is not isolated. There exists a sequence $\{h_k\}$ of distinct points of H converging to 0. For each k let m_k be the least positive integer such that $\|m_k h_k\| < R$ where R is such that $\sup\{\|h_k\| : k = 1, 2, ...\} \leq R$. Then $\{m_k h_k\} \subset \Delta$ where $\Delta = \{y \in \mathbb{R}^n : \|y\| < R\}$. Note that $(m_k + 1)h_k \notin \Delta$. Claim: there exists $k_j \uparrow \infty$ and $y_0 \in \mathbb{R}^n$ such that $m_{k_j} h_{k_j} \to y_0$ and $(m_{k_j}+1)h_{k_j} \to y_0$. Since $\{m_k h_k\} \subset \Delta$ there exists $k_j \uparrow \infty$ and $y_0 \in \mathbb{R}^n$ such that $m_{k_j} h_{k_j} \to y_0$. Since $h_k \to 0$ we see that $(m_{k_j} + 1)h_{k_j} \to y_0$ too. The claim is proved. It follows that $y_0 \in \partial\Delta$ and hence $y_0 \neq 0$. Note that $y_0 \in H$ because H is closed and $\{m_k h_k\} \subset H$. Let $L = \{ty_0 : t \in \mathbb{R}\}$. If $t \in \mathbb{R}$ then $\left\|[tm_{k_j}]h_{k_j} - ty_0\right\| \leq \left\|[tm_{k_j}]h_{k_j} - tm_{k_j}h_{k_j}\right\| + \left\|tm_{k_j}h_{k_j} - ty_0\right\| \leq \left\|h_{k_j}\right\| + |t| \left\|m_{k_j}h_{k_j} - y_0\right\|$ [We used the fact that $|[x] - x| \leq 1$ for any real number x]. Since $h_{k_j} = (m_{k_j} + 1)h_{k_j} - m_{k_j}h_{k_j} \to y_0 - y_0 = 0$ it follows that $[tm_{k_j}]h_{k_j} \to ty_0$ which imlies that $ty_0 \in H$. Thus $H \cap L = L$.

Problem 366

Show that there is no meaurable function $f : S^1 \to \mathbb{R}$ such that $f(ab) = f(a) + f(b)$ $\forall a, b \in S^1$ and f not identically 0. Does there exist a (non-measurable) function $f : S^1 \to \mathbb{R}$ such that $f(ab) = f(a) + f(b)$ $\forall a, b \in S^1$ and f not identically 0?

For the first part define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(e^{it})$. Then g is a measurable additive function on $\mathbb{R}$ and hence there exists a real number c such that $f(e^{it}) = g(t) = ct$ for all t. Since $f(e^{it})$ is periodic we get $c = 0$ and $f \equiv 0$. We now prove the existence of a function $f : S^1 \to \mathbb{R}$ such that $f(ab) = f(a) + f(b)$ $\forall, b \in S^1$ and f not identically 0. Let H be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. Let $t_0, t_1, ...$ be a convergent sequence of distinct points in H. Such a sequence exists because H is uncountable. Let $\xi : H \to \mathbb{R}$ be any function such that $\xi(t_j) = j$ for $j \geq 0$. Let $g : \mathbb{R} \to \mathbb{R}$ be obtained by linearly extending ξ. Let $f(e^{it}) = g(\frac{t_0}{2\pi}t)$. Note that $e^{it} = e^{is}$ implies $t = s + 2n\pi$ for some integer n and $g(\frac{t_0}{2\pi}t) = g(\frac{|t|}{2\pi}s)$ because g is additive and $g(\frac{t_0}{2\pi}2n\pi) = ng(t_0) = n\xi(t_0) = 0$. Hence f is well defined. Clearly f satisfies the functional equation $f(ab) = f(a) + f(b)$ $\forall a, b \in S^1$.

Remarks. there exists $f : \mathbb{C}\backslash\{0\} \to \mathbb{R}$ such that $f(ab) = f(a) + f(b)$. Simply compose the map constructed above with the map $z \in \frac{z}{|z|}$. This map cannot coincide with any branch of the logairthm obtained by deleting a ray through the origin because these branches are measurable and f is not. If $f : \mathbb{C}\backslash\{0\} \to \mathbb{R}$ is a measurable function such that $f(ab) = f(a)+f(b)$ for all a, b then $f(a) = c \log |a|$ for some $c \in \mathbb{R}$.

Problem 367

Let $f \in C(S^1)$ and μ be the normalized Haar measure on S^1. Show that $\int f d\mu = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} f(c^k)$ for any $c \in S^1$ which is not a root of unity. This is immediate from The Ergodic Theorem. Give a proof without using The Ergodic Theorem.

By Stone-Weierstras Theorem $\{f_j : j \in \mathbb{Z}\}$, where $f_j(z) = z^j$ spans a dense subspace of $C(S^1)$. Hence it suffices to prove the result when $f = f_j$ for some integer j. In this case $\int f d\mu = 0$ if $j \neq 0$ and 1 if $j = 0$. Let $c = e^{ia}$. Then $\frac{a}{2\pi}$ is irrational. Clearly $\frac{1}{2n+1} \sum_{k=-n}^{n} f(c^k) = \frac{1}{2n+1} \sum_{k=-n}^{n} c^{jk} = \frac{1}{2n+1} \sum_{k=-n}^{n} e^{ijka}$. Since $\sum_{k=-n}^{n} e^{ikt} = \sum_{k=0}^{n} e^{ikt} + \sum_{k=-n}^{-1} e^{ikt} = \frac{1-e^{i(n+1)t}}{1-e^{it}} + \frac{e^{-it}-e^{-(n+1)t}}{1-e^{-it}}$ which is bounded (if $\cos t \neq 1$ we see that $\frac{1}{2n+1} \sum_{k=-n}^{n} f(c^k) \to 0$ if $\cos ja \neq 1$ which is true since $\frac{a}{2\pi}$ is irrational.

Remark: a more general result is the following: let G be a compact metric group and $G_n, n = 1, 2, ...$ be an increasing sequence of closed subgroups whose union is dense in G. Let μ, μ_n be the Haar measures on G, G_n respectively. Suppose characters on G span a dense subsapce of $C(G)$. Then $\int f d\mu = \lim \int f d\mu_n$ for all f in $C(G)$. In this case the character group is a countable orthonormal sequence and it suffices to prove that result when f is a character. In this case both sides of the equation $\int f d\mu = \lim \int f d\mu_n$ are 0 according if $f \neq 1$ and 1 if $f = 1$.

Problem 368

Let $(X, \tau, *)$ be a compact Hausdorff group which is also a topological semigroup. If left and right cancellation laws hold in X show that X is a topological group.

By Zorn's Lemma there is a smallest non-empty closed set C such that $C * X \subset C$. Note that $C * X$ is closed, non-empty and $(C * X) * X \subset C * X$. By minimality of C we get $C = C * X$. Let $c \in C$. We claim that $c * X = C$. First note that $(c * X) * X \subset c * X$ and $c * X \subset C * X \subset C$. Minimality of C shows $C = c * X$. Now, if $x \in X$ then $c * x * X = C$ because $c * x * X \subset C$ and

$c*x*X*X \subset c*x*X$. Thus $c*x*X = C = C*X$. From this it follows that $x*X = X$. [$y \in X$ implies $c*y \in C*X = c*x*X$ so $c*y = c*x*z$ for some z which implies $y = x*z \in x*X$ proving that $X \subset x*X$]. From this it follows by standard arguments that S is a group. $x_i \to x$ implies $x_i * x^{-1} \to e$ and $x_i^{-1} * x_i * x^{-1} \to z*x*x^{-1} = z$ along a subnet with $x_i^{-1} \to z$. Thus $x^{-1} = z$ proving that the only accumulation point of $\{x_i^{-1}\}$ is x^{-1}. Hence $x_i^{-1} \to x^{-1}$ and X is a topological group.

Problem 369

Does there exists a probability measure μ on the Borel sigma field of $\mathbb{R}^\infty$ which is translation invaraint?

No! Let $p_1, p_2, ...$ be the projection maps. Choose positive numbers $c_1, c_2, ...$ such that $\mu\{|p_j| > \frac{1}{2^j c_j}\} < \frac{1}{2^j}$. Then $\mu\{|p_j| > \frac{1}{2^j c_j}$ infinitely often$\} = 0$ and $\sum c_j p_j$ converges almost surely. Let $M = \{(x_j) \in \mathbb{R}^\infty : \sum c_j x_j$ converges$\}$. M is a proper linear subspace and $\mu(M) = 1$. If $x \notin M$ then $x + M$ and M are disjoint and these sets are also Borel sets. Hence they cannot have the same measure.

Problem 370

Generalize Problem 369 by showing that no Borel probability measure on a separable infinite dimensional Frechet space (over $\mathbb{R}$) can be translation invariant.

Let μ be a Borel probability measure on a separable infinite dimensional Frechet space X. There exist compact sets $K_n, n = 1, 2, ...$ such that $\mu(K_n) > 1 - \frac{1}{n}$. We may suppose $K_n \subset K_{n+1}$. Let M_n be the space spanned by K_n. Then $M_n = \bigcup_{k=1}^{\infty} \{\sum_{i=1}^{k} a_i x_i : a_i's \in \mathbb{R}, x_i's \in K_n, |a_i| \le k \ \forall i\} = \bigcup_{k=1}^{\infty} M_{n,k}$ where $M_{n,k} = \{\sum_{i=1}^{k} a_i x_i : a_i's \in \mathbb{R}, x_i's \in K_n, |a_i| \le k \ \forall i\}$. Note that $M_{n,k}$ is compact. [It is a continuous image of $[-k, k]^k \times K_n^k$]. Now $M \equiv \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} M_{n,k} = \bigcup_{n=1}^{\infty} M_n$ is a linear subspace of X. It is a proper subspace by Baire Category Theorem. [Each $M_{n,k}$ is closed and has empty interior.(By Theorem 1.22 of Rudin's Functional Analysis any locally compact t.v.s. is of finite dimension. If $M_{n,k}$ has nonempty interior then, by translation, there is a compact set H such that $0 \in H^0$. But then $X = \bigcup_{n=1}^{\infty} nH^0$ which makes X locally compact]. Hence $\mu(M) = 1$ and M is a proper subspace which implies that M and $M + x$ cannot have the same measure when $x \notin M$ (as in previous problem).

Problem 371

Show that we cannot have identically distributed random variables X and Y such that $X < Y$ almost surely.

Let F be the common distribution. For any real number x we have $P\{Y \leq x, X < Y\} = P\{Y \leq x\} = F(x)$ and $P\{Y > x, x < X < Y\} = P\{X > x\} = 1 - F(x)$. Since $\{X \leq x < Y\}^c$ contains the disjoint union $\{Y > x, x < X < Y\} \cup \{Y \leq x, X < Y\}$ we get $1 - P\{X \leq x < Y\} \geq 1 - F(x) + F(x) = 1$. Hence $P\{X \leq x < Y\} = 0$ for every x. This is a contradiction.

Problem 372 [A characterization of conditional expectation operators]

Let $T : L^1 \to L^1$ be a continuous linear map with $\|T\| = 1, T1 = 1$ and $T(YT(X)) = (TY)(TX)$ for all $X \in L^1, Y \in L^\infty$. [L^1 stands for $L^1(\Omega, \mathcal{F}, P)$ where $(\Omega, \mathcal{F}, P)$ is a probabilty space]. Then there exists a sigma field $\mathcal{G}$ contained in $\mathcal{F}$ such that $TX = E(X|\mathcal{G})$ for all $X \in L^1$. The converse is also true.

The converse part follows from standard facts about conditional expectations. Suppose now that T has the stated properties. Let $M = \{X \in L^\infty : TX = X\}$. Let $\mathcal{G}$ be the sigma field generated by M (the smallest one which makes each X in M measurable). Claim: $Y \in L^\infty$ implies $TY \in L^\infty$. For this we first verify that $[T(Y)]^n \in L^1$ for each positive integer n. Indeed, Y and $TY \in L^1$ and if $[T(Y)]^k \in L^1$ then $[T(Y)]^{k+1} = [TY][T(Y)]^k = T[Y[T(Y)]^k] \in L^1$ because $Y[T(Y)]^k \in L^1$. This proves that $[T(Y)]^n \in L^1$ for each positive integer n. Now $\|TY\|_n^n = \int |T(Y)|^n = \int \left|T[Y(TY)^{n-1}]\right| \leq \int \left|Y(TY)^{n-1}\right| \leq [\int |Y|^n]^{1/n}[\int |TY|^n]^{1-1/n}$ which shows $\int |T(Y)|^n \leq \int |Y|^n$. So $\|TY\|_n \leq \|Y\|_\infty$. Letting $n \to \infty$ we get $\|TY\|_\infty \leq \|Y\|_\infty$ proving the claim. Now suppose $X \in M$. We claim that $T(X^n) = X^n$ for all n. Indeed, if this holds for $n = k$ then $T(X^{k+1}) = (TX)(TX^k) = X(X^k) = X^{k+1}$. It follows that $T(p(X)) = p(X)$ for any polynomial p. Approximating any continuous function on $[-\|X\|_\infty, \|X\|_\infty]$ by polynomials we see that $T(f(X)) = f(X)$ for any continuous function f. From this it follows that $T(f(X)) = f(X)$ for any bounded $\mathcal{B}(\mathbb{R})$ meaurable f. [There exist a sequence of continuous functions $\{f_n\}$ such that $\int |f_n - f| \, dP \circ X^{-1} \to 0$. Since $Tf_n \to Tf$ in $L^1(P \circ X^{-1})$ we get $Tf_n(X) \to Tf(X)$ in L^1. There is a subsequence $\{n_j\}$ of the integers such that $Tf_{n_j}(X) \to Tf(X)$ a.s. and $f_{n_j}(X) \to f(X)$ a.s. Since $Tf_{n_j}(X) = f_{n_j}(X)$ for each j we get $T(f(X)) = f(X)$]. We conclude that $TY = Y$ for any $\mathcal{G}$ meaurable Y. [$\{E \in \mathcal{G} : TI_E = I_E\}$ is a sigma field because $T1 = 1$ and $E_n \downarrow E$ implies $TI_{E_n} \to TI_E$ in L^1. (Note that $TI_E = I_E$ and $TI_F = I_F$ imply $TI_{E \cap F} = T(I_E I_F) = T(I_E T(I_F)) = (TI_E)(TI_F) = I_E I_F = I_{E \cap F}$). This sigma field contains $X^{-1}(A)$ for any Borel set A in $\mathbb{R}$ because $TI_{X^{-1}(A)} =$

$TI_A(X) = I_A(X) = I_{X^{-1}(A)}$. Hence it contains the sigma field genertaed by such sets which is $\mathcal{G}$]. Of course $T(T(X)) = T(1(T(X)) = (T1)T(X) = T(X)$ so $T^2 = T$. Hence $T(X)$ is $\mathcal{G}$ measurable for any $X \in L^1$. Now for $X \in L^\infty$ and $E \in \mathcal{G}$ we have $I_E T(X) = T(I_E)(TX) = T(XT(I_E)) = T(XI_E)$ so $\int_E T(X) = \int I_E T(X) = \int T(XI_E)$ To complete the proof we show that $\int T(X) = \int X$ for any $X \in L^\infty$. This would give $\int_E T(X) = \int T(XI_E) = \int XI_E = \int\limits_E X$ proving that $TX = E(X|\mathcal{G})$. Consider the adjoint operator $T^* : L^\infty \to L^\infty$. We have $1 = \int 1T(1) = \int [T^*(1)]1 \leq \|T^*1\| \leq \|T^*\| = \|T\| = 1$ so equality holds throughout. Hence $T^*1 = 1$ which gives $\int T(X) = \int XT^*(1) = \int X$ for any $X \in L^\infty$.

Problem 373

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}$ be a sub-sigma field of $\mathcal{F}$. Let $M = \{E(X|\mathcal{G}) : X \in L^2\}$. Show that, $1 \in M$, M is a closed subspace of L^2 and that $\max\{f, g\} \in M$ whenever f and $g \in M$. Prove that any subspace M with these properties coincides with $\{E(X|\mathcal{G}) : X \in L^2\}$ for some sub-sigma $\mathcal{G}$ field of $\mathcal{F}$.

Firts part is trivial since $\{E(X|\mathcal{G}) : X \in L^2\}$ is nothing but the set of all those elements X of L^2 which are $\mathcal{G}$ measurable. [The fact that $E(X|\mathcal{G}) \in L^2$ follows by Jensen's inequality for conditional expectations]. Now suppose M has the stated properties. Let $\mathcal{G} = \{E \in \mathcal{F} : I_E \in M\}$. If $I_E \in M$ and $I_F \in M$ then $I_{E \cup F} = \max\{I_E, I_F\} \in M$. Since $1 \in M$ it follows immediately that $\mathcal{G}$ is a field. If $E_n \uparrow E$ and each $E_n \in \mathcal{G}$ then I_E is the L^2 limit of $\{I_{E_n}\} \subset M$ and hence $E \in \mathcal{G}$. Thus $\mathcal{G}$ is a sigma field. Since every simple function in $L^2(\Omega, \mathcal{G}, P)$ belongs to M and M is closed it follows that every function in $L^2(\Omega, \mathcal{G}, P)$ belongs to M. Let $Y \in M$ and $a \in \mathbb{R}$. Then $-\max\{-1, \min\{0, n(Y-a)\}\} \in M$ and $-\max\{-1, \min\{0, n(Y-a)\}\} \to \begin{cases} 0 \text{ if } Y \geq a \\ 1 \text{ if } Y < a \end{cases}$. It follows, by Dominated Convergence Theorem that $I_{\{Y<a\}} \in M$. Hence $\{Y < a\} \in \mathcal{G}$ for every real number a. Hence Y is $\mathcal{G}-$ measurable.

Problem 374

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}$ be a sub-sigma field of $\mathcal{F}$. Let $X \in L^1$ and suppose X and $E(X|\mathcal{G})$ have the same distribution. Show that $X = E(X|\mathcal{G})$ a.s..

Let $Y = X^+$. We claim that Y and $E(Y|\mathcal{G})$ have the same distribution. Indeed, Y has the same distribution as $(E(X|\mathcal{G}))^+$ and $(E(X|\mathcal{G}))^+ \leq E(Y|\mathcal{G})$ with both sides having the same expectation. Hence $(E(X|\mathcal{G}))^+ = E(Y|\mathcal{G})$ a.s. which implies that $Y = X^+$ and $E(Y|\mathcal{G})$ have the same distribution. Since $-X$ and $E(-X|\mathcal{G})$ have the same distribution we see that $(-X)^+$ and $E((-X)^+|\mathcal{G})$ have the same distribution. This means $(X)^-$ and $E((X)^-|\mathcal{G})$ have the same distribution. If we prove the result for non-negative random variables we conclude that $X^+ = E(X^+|\mathcal{G})$ a.s.. and $X^- = E(X^-|\mathcal{G})$ a.s.. and hence that $X = E(X|\mathcal{G})$ a.s.. From now on we assume that $X \geq 0$. Let N be a positive integer and $X_N = \min\{X, N\}$. We calim that X_N and $E(X_N|\mathcal{G})$ have the same distribution. For this note that $E(X_N|\mathcal{G}) \leq \min\{E(X|\mathcal{G}), N\}$ and both sides have the same expectation so equality holds a.s.. Since $\min\{E(X|\mathcal{G}), N\}$ has the same distribution as $\min\{X, N\} = X_N$ we see that X_N and $E(X_N|\mathcal{G})$ have the same distribution. If we prove the result for non-negative bounded random variables we can conclude that $E(X_N|\mathcal{G}) = X_N$ a.s.. This is true for each N and we get $E(X|\mathcal{G}) = X$ a.s. in the limit. We now assume that X is positive and bounded. In this case $(E(X|\mathcal{G}))^2 \leq E(X^2|\mathcal{G})$ and both sides have the same expectation. Hence $(E(X|\mathcal{G}))^2 = E(X^2|\mathcal{G})$ a.s.. This gives $E(X - E(X|\mathcal{G}))^2 = EX^2 + EX^2 - 2E\{XE(X|\mathcal{G})\} = 2EX^2 - 2E\{E(X|\mathcal{G})\}^2 = 0$ and so $X = E(X|\mathcal{G})$ a.s.

Remark: compare with the following fact: if M is a closed subspace of a Hilbert space H and P is the projection onto M then $\|x\| = \|Px\|$ implies $x = Px$. This is trivial and this gives above result when $X \in L^2$. Clearly the 'full force ' of the hypothesis is not required in above proof.

Problem 375

Suppose $\mathcal{G}_1$ and $\mathcal{G}_1$ are sub sigma fields of $\mathcal{F}$ where $(\Omega, \mathcal{F}, P)$ is a given probabilty space. Suppose $X - E(X|\mathcal{G}_1) = E(X|\mathcal{G}_2)$ whenever $X \in L^1(\Omega, \mathcal{G}, P)$ and $EX = 0$ where $\mathcal{G}$ is the sigma field generated by $\mathcal{G}_1$ and $\mathcal{G}_2$. Show that at least one of the sigma fields $\mathcal{G}_1$ and $\mathcal{G}_1$ is trivial w.r.t. P.

Motivation: if T is the projection of a Hilbert space onto a closed subspace then $I - P$ is always a projection.

Let $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$. Putting $X = I_{A\cup B} - P(A \cup B)$ we have $I_{A\cup B} - E(I_A + I_B - I_{A\cap B}|\mathcal{G}_1) = E(I_A + I_B - I_{A\cap B}|\mathcal{G}_2) - P(A \cup B)$. Hence $I_{A\cup B} - I_A - E(I_B|\mathcal{G}_1) + I_A E(I_B|\mathcal{G}_1) = E(I_A|\mathcal{G}_2) + I_B - I_B E(I_A|\mathcal{G}_2) - P(A \cup B)$. If we prove that $\mathcal{G}_1$ and $\mathcal{G}_1$ are independent we can conclude that $I_{A\cup B} - I_A - P(B) + I_A P(B) = P(A) + I_B - I_B P(A) - P(A \cup B)$. If $P(A\backslash B) > 0$ we can evaluate both sides on $A\backslash B$ to get $1 - 1 - P(B) + P(B) = P(A) + 0 - 0 - P(A \cup B)$ or $P(A\cup B) = P(A)$. This means $P(B\backslash A) = 0$. We have proved that $P(A\backslash B) = 0$ or $P(B\backslash A) = 0$. Independence of A and B no shows that $P(A)$ and $P(B)$ cannot both belong to $(0, 1)$. [Note that if one of the sigma fields $\mathcal{G}_1$ and $\mathcal{G}_1$ is trivial then the stated identity indeed holds]. We now prove that $\mathcal{G}_1$ and $\mathcal{G}_1$ are independent. Put $X = I_A - P(A)$ where $A \in \mathcal{G}_1$. We get $0 = E(I_A|\mathcal{G}_2) - P(A)$. Integrating over any $B \in \mathcal{G}_2$ we get $P(A \cap B) = P(A)P(B)$.

Problem 376

Let T be a unitary operator on $\mathbb{C}^N$ such that for every positive integer n there exists an positive integer k such that T^{kn} is similar to T, i.e. there is invertible operator S such that $T^{kn} = S^{-1}TS$. Show that $T = I$.

There is a basis consisting of eigen values so it suffices to show that 1 is the only eigen value of T. Let $\{\lambda_1, \lambda_2, ..., \lambda_N\}$ be the set of eigen values of T. Then the equation $T^{kn} = S^{-1}TS$ shows that $\{\lambda_1, \lambda_2, ..., \lambda_N\} = \{\lambda_1^{kn}, \lambda_2^{kn}, ..., \lambda_N^{kn}\}$. Each λ_i is of the type λ_j^l for infinitely many l (for some j) which implies that λ_j and hence λ_i is a root of unity. There exists an integer m such that $\lambda_i^m = 1$ for each i. But then there exists k such that $\{\lambda_1, \lambda_2, ..., \lambda_N\} = \{\lambda_1^{mn}, \lambda_2^{km}, ..., \lambda_N^{km}\}$ and since $\lambda_j^{km} = 1$ for all j we see that $\lambda_i = 1$ for all i.

Problem 377

Let G be a compact abelian group and H be a subgroup of the dual group $G\hat{}$. If $\gamma \in G\hat{}\backslash H$ and $\delta \in M$ where M is the space of all finite linear combinations of elements of H show that $L^2(m)$ distance between γ and δ is not less than 1. [m is the normalized Haar meaure on G]. Hence show that the only subgroup of $G\hat{}$ which separates points of G is $G\hat{}$ itself.

Let $\delta = \sum_{j=1}^{k} c_j \gamma_j$ with $\gamma_j' s \in H$. We claim that $\int \gamma \bar{\gamma_j} dm = 0$ for each j. This is because $\gamma \bar{\gamma_j}$ is a character γ_0 not identically equal to 1 and $\gamma_0(g_0) \int \gamma_0(g) dm(g) = \int \gamma_0(g_o g) dm(g) = \int \gamma_0(g) dm(g)$ so $\int \gamma_0(g) dm(g) = 0$. For the same reason $\gamma_j' s$ are orthogonal to each other (assuming, of course, that they are distinct). Now $\|\gamma - \delta\|_2^2 = 1 + \sum_{j=1}^{k} |c_j|^2 \geq 1$. The second part now follows by an easy application of Stone-Weierstrass Theorem: let $C_b(G)$ be the Banach space of bounded continuous complex functions on G with the supremum norm. If H separates points so does M, which is a subalgebra of $C_b(G)$. Further M contains constants and it is closed under conjugation. Hence M is dense in $C_b(G)$. This contradicts the first part if there is an element γ in $G\hat{}\backslash H$.

Problem 378

Let μ be a complex Borel measure on $\mathbb{R}$ such that $\mu(x + E) \to \mu(E)$ as $x \to 0$ whenever E is a compact set whose Lebesgue measure is 0. Show that μ is absolutely continuous w.r.t. Lebesgue measure.

By regularity of μ it suffices to show that $\mu(E)=0$. Let $d\nu(x)=I_{(-\delta,\delta)}(x)dx$. Then $(\nu*|\mu|)(E)=\int\nu(E-x)d|\mu|(x)=0$ since $\nu(E-x)=0$ for all x. Hence $0=\frac{1}{2\delta}\int|\mu|(E-x)d\nu(x)=\frac{1}{2\delta}\int|\mu|(E-x)I_{(-\delta,\delta)}(x)dx\geq\frac{1}{2\delta}\int|\mu(E-x)|I_{(-\delta,\delta)}(x)dx\to$ $|\mu(E)|$ as $\delta\to 0$.

Problem 379

Let M be a closed subspace of a real Hilbert space H and $x\in H$ with $\alpha\equiv d(x,M)>0$. For any $m_1,m_2\in M$ prove that $\|m_1-m_2\|\leq\{\|x-m_1\|^2-\alpha^2\}^{1/2}+\{\|x-m_2\|^2-\alpha^2\}^{1/2}$

We have $\left\|x-\frac{1}{1-c}(m_1-cm_2)\right\|^2\geq\alpha^2$ if $c\neq 1$. Hence $\|(x-m_1)+c(m_2-x)\|^2\geq|1-c|^2\alpha^2$ and this last inequality holds for $c=1$ also. Thus $\|x-m_1\|^2+c^2\|x-m_2\|^2-2c<x-m_1,x-m_2>\geq(1-c)^2\alpha^2$. The validity of this for all c implies that $[<x-m_1,x-m_2>-\alpha^2]^2\leq[\|x-m_1\|^2-\alpha^2][\|x-m_2\|^2-\alpha^2]$. [Take $c=\frac{<x-m_1,x-m_2>-\alpha^2}{\|x-m_2\|^2-\alpha^2}$]. Finally, $\|m_1-m_2\|^2=\|(x-m_1)-(x-m_2)\|^2\leq\|x-m_1\|^2+\|x-m_2\|^2-2<x-m_1,x-m_2>$
$\leq[\|x-m_1\|^2-\alpha^2]+[\|x-m_2\|^2-\alpha^2]+2[\|x-m_1\|^2-\alpha^2]^{1/2}[\|x-m_2\|^2-\alpha^2]^{1/2}$ which gives the desired inequality.

Problem 380

Let $K\subseteq\mathbb{C}$ be a compact set such that the unbounded component of K^c contains 0. Show that there is a simply connected open set Ω such that $K\subseteq\Omega$ and there is an analytic branch of logarithm in Ω.

Let $c_n\to\infty,c_n$ belonging to the unbounded component C of K^c. Let $c_0=0$. Since there are continuous paths from 0 to c_1,c_1 to c_2 etc we can find a continuous map $\gamma:[0,1)\to C$ such that $\gamma(1-\frac{1}{n})=c_n$ for all n and γ is linear in $[1-\frac{1}{n},1-\frac{1}{n+1}]$ for each n . We claim that $D\equiv\gamma[0,1)\cup\{\infty\}$ is connected in the extended plane. If we can write D as the disjoint union of non-empty open subsets U and V then the connected set $\gamma[0,1)$ is contained in either U or V. Suppose it is contained in U. Then $\infty\in V$. Now $\gamma[0,1)\subseteq U\subseteq V^c$ and V^c is closed. Since $\infty=\lim c_n=\lim\gamma(1-\frac{1}{n})$ it follows that $\infty\in V^c$, a contradiction. This proves the claim. Let $\Omega=D^c$. Then Ω is open, [If $\{\gamma(t_k)\}$ converges in $\mathbb{C}$ either $\{t_k\}\to 1$ or $\{t_k\}$ has a subsequence converging to a point t of $[0,1)$. In the first case $\lim\gamma(t_k)=\infty$ and in the second case $\lim\gamma(t_k)=\gamma(t)$]. Clearly $K\subseteq\Omega$. Since $0\notin\Omega$ we only have to show that Ω is simply connected. Its complemenet in the extended complex plane is D which is connected.

Problem 381

Let $g:[0,1]\to[0,1]$ be a continuous map, $X=C([0,1])$ and define $T:X\to X$ by $Tf(x)=f(g(x))$. For what continuous functions g is this map compact?

We prove that T is compact if and only if g is a constant. If g is a constant then T is compact because $[0,1]$ is compact. Suppose T is compact. Claim: whenever $t_n \to t$ in $[0,1]$ we have $g(t_n) = g(t)$ for infinitely many n. If the claim is false there exists $t_n \to t$ with $g(t_n) \neq g(t)$ for all n sufficiently large, say $n \geq k$. There exists functions $f_n, n = k, k+1, ...$ in X such that $0 \leq f_n \leq 1, f_n(g(t_n)) = 0$ and $f_n(g(t)) = 1$. Then $|Tf_n(t_n) - Tf_n(t)| = 1$ for all $n \geq k$ which implies that $\{Tf_{n_j}\}$ is not equi-continuous in X whenever $n_j \uparrow \infty$. Thus $\{Tf_n\}$ has no convergent subsequence which implies that T is not compact. We have now proved the claim. To see why g must be a constant we just have to observe that $\{t : g(t) = g(0)\}$ is open and closed in $[0,1]$. [If $g(t_n) \neq g(0)$ for all n and $t_n \to t$ then $g(t) \neq g(0)$ by the claim so $\{t : g(t) \neq g(0)\}$ is closed. Of course $\{t : g(t) = g(0)\}$ is closed by continuity].

Problem 382

Prove or disprove that any compact operator on $X \equiv L^p([0,1])$ (where $1 \leq p < \infty$) is a limit (in operator norm) of finite rank operators.

True. We first make some preliminary observations.

Fact 1: if $T, T_n, n = 1, 2, ...$ are bounded operators on a Banach space X such that $\|T_n x - Tx\| \to 0$ for each x then $\|T_n x - Tx\| \to 0$ uniformly on compact subsets of X. This follows from triangle inequality and the fact that $\{\|T_n\|\}$ is bounded. [For a convergent sequence $\{x_n\}$ in the given compact set $\|T_n x_n - Tx_n\| \leq \{\sup \|T_n\|\} \|x_n - x\| + \|T_n x - Tx\| + \|T\|\} \|x_n - x\|$ so $T_n x_n - Tx_n \to 0$].

Fact 2: if $\{S_n\}$ is a sequence of operators such that $\|S_n x - x\| \to 0$ for each x and if T is a compact operator on X then $\|S_n T - T\| \to 0$. This is immediate from Fact 1.

Fact 3: let $\mathcal{G}_n$ be the sigma field generated by the sets $[\frac{i-1}{2^n}, \frac{i}{2^n}), 1 \leq i < 2^n$. Let $S_n f = E(f|\mathcal{G}_n)$. Then the operators S_n satisfy the hypothesis of Fact 2 when $X \equiv L^p([0,1])$.

This is a standard result using uniform integrability of $\{E(f|\mathcal{G}_n)\}$ for fixed f.

It follows from Fact 2 that if T is compact then the finite rank operators $S_n T$ converge to T in operator norm.

Problem 383

Let T be a bounded operator on a Hilbert space H such that $T^2 = T$ and $\|T\| = 1$. Show that T is the projection onto its range.

Note that $T(H) = \{x \in H : Tx = x\}$. Hence $T(H)$ is closed. Let P be the projection with range $T(H)$. We have to show that $x - T(x) \perp T(H)$ for which it suffices to show that $T(H) \subseteq [\ker(T)]^{\perp}$. Let $y \in T(H)$ and write y as $y_1 + y_2$ with $y_1 \in \ker(T)$ and $y_2 \in [\ker T]^{\perp}$. If we show that $[\ker T]^{\perp} \subseteq T(H)$ it would follow that $y_1 = y - y_2 \in T(H)$ and this implies $y_1 = 0$ and $y = y_2 \in [\ker T]^{\perp}$, as required. It reamins to show that $[\ker T]^{\perp} \subseteq T(H)$. Let $x \in [\ker T]^{\perp} \backslash \{0\}$. Then $0 =< x, x - Tx >= \|x\|^2 - < x, Tx >$ so $\|x\|^2 =< x, Tx > \leq \|x\| \|Tx\| \leq \|x^2\|$ which implies $\|Tx\| = \|x\|$. Since $< x, Tx >= \|x\|^2$ we get $\|x - Tx\|^2 = \|x\|^2 + \|Tx\|^2 - 2\|x\|^2 = 0$ and $x = Tx \in T(H)$.

Problem 384

If T is a compact operator on a Hilbert space H with an orthonormal basis $\{e_n\}$ show that $Te_n \to 0$ but the converse is false.

If T is of finite rank than $Tx = \sum_{j=1}^{N} < x, x_j > y_j$ for suitable $N, x_j, y_j, 1 \leq j \leq N$, so $Te_n \to 0$. The general case follows from the fact that $\|T - T_n\| \to 0$ for some sequenec $\{T_n\}$ of finite rank opeartors. For the second part define $T : l^2 \to l^2$ by $Te_n = n^{-1/2}e_1$ where $\{e_n\}$ is the standard basis of l^2. Let $x_N = \sum_{j=1}^{N} a_j e_j$ where $a_j = j^{-1/2}/a$ and $a = (\sum_{j=1}^{N} j^{-1})^{1/2}$. Then $\|x_N\|^2 = 1$ for all N and $T(x_N) = (\sum_{j=1}^{N} a_j j^{-1/2})e_1 = \sqrt{\sum_{j=1}^{N} \frac{1}{j}} e_1$. Thus T is not compact.

Problem 385 [Wilansky]

Let X be a real normed linear space and $T : X \to X$ be an additive map. If $\sup\{\|Tx\| : \|x\| < 1\} < \infty$ show that T is a bounded linear map. What happens if the hypothesis $\sup\{\|Tx\| : \|x\| < 1\} < \infty$ is changed to $\sup\{\|Tx\| : \|x\| = 1\} < \infty$?

Fix x with $\|x\| < 1$ and define $\phi : \mathbb{R} \to X$ by $\phi(a) = T(ax) - aT(x)$. ϕ is an additive map so $\phi(ra) = r\phi(a)$ if r is rational. Since $\phi(1) = 0$ we get $\phi(r) = 0$ for all r rational. Note also that $\|\phi(a)\| < 2M$ for $0 \leq a < 1$ where M is the supremum in the statement of the problem. If r is a positive rational and a is any real number we can find a rational s such that $0 \leq ra + s < 1$. We now have $\|\phi(ra + s)\| < 2M$. But $\phi(ra + s) = r\phi(a) + \phi(s) = r\phi(a) + 0$ so $\|\phi(a)\| < 2M/r$. Letting $r \to \infty$ we get $\phi(a) = 0$ which gives $T(ax) = aT(x)$ if $a \in \mathbb{R}$ and $\|x\| < 1$. Now let $y \in X$ and $a \in \mathbb{R}$. We have $T(ay) = T(\{2a\|y\|\}\frac{y}{2\|y\|}) = 2a\|y\| T(\frac{y}{2\|y\|}) = aT(\{2\|y\|\}\frac{y}{2\|y\|}) = aT(y)$. We have proved that T is a linear map and boundedness follows immediately. This is Wilansky's proof. Here is an alternative proof: fix $x_0 \neq 0$ and $x^* \in X^*$ and consider the map $a \to x^*(T(ax_0)) - ax^*(T(x_0))$. This map from $\mathbb{R}$ into itself is additive and vanishes at 1. It is bounded on $\{a : |a| < 1/\|x_0\|\}$. These facts imply that it vanishes identically. It follows that $x^*(T(ax_0)) = ax^*(T(x_0))$. Since x^* is arbitrary this gives$T(ax_0) = aT(x_0)$ $\forall a \in \mathbb{R}$. Thus T is linear and bounded.

The case $X = \mathbb{R}$ shows that T additive and $\sup\{\|Tx\| : \|x\| = 1\} < \infty$ does not imply that T is linear.

Problem 386

Let M be a closed subspace of a real Banach space X and $\pi : X \to X|M$ be the quotient map. When is π a closed map?

We claim that π is not closed if $M \neq \{0\}$. [If $M = \{0\}$ then π is closed]. Let x_1 be a unit vector in M and x_2 be a unit vector not in M. Let $C = \{ax_1 + bx_2 : a, b \in \mathbb{R}, ab = 1\}$. Using linear independence of x_1 and x_2 it is easy to see that C is closed. [$ax_1 + bx_2 \to a$ and $ax_1 + bx_2 \to b$ are well-defined linear maps on a finite dimensional space, hence continuous]. Note that $\pi(C) = \{bx_2 + M : b \neq 0\}$ which is not closed since it contains $\{\frac{1}{n}x_2 + M\}$ which converges to the zero element of $X|M$ which does not belong to $\pi(C)$.

Problem 387

Let X be a compact Hausdorff space and Y be a closed susbset of X. Let $M = \{f \in C(X) : f = 0 \text{ on } Y\}$. Show that $C(X)|M$ is isometrically isomorphic to $C(Y)$.

Define $T : C(X)|M \to C(Y)$ by $T(f + M) = f_Y$ where f_Y is the restriction of f to Y. If $f \in M$ then $f_Y = 0$ so T is a well-defined linear map. Tietze Extension Theorem shows that T is onto. T is obviously one-to-one. We now show that $\|f + M\| = \|f_Y\|$. Since $\|f + g\| \geq \sup\{|f(x) + g(x)| : x \in Y\} = \|f_Y\|$ for all $g \in M$ it follows that $\|f + M\| \geq \|f_Y\|$. Now let $\varepsilon > 0$ and $U = \{x : |f(x)| < \|f_Y\| + \varepsilon\}$. U is open and $Y \subseteq U$. There exists a continuous function $g : X \to [0, 1]$ such that $g = 0$ on Y and $g = 1$ on U^c. Now $\|f + M\| \leq \|f + g\|$. On Y $|f(x) - f(x)g(x)| \leq \|f_Y\|$. If $x \in U^c$ then $|f(x) - f(x)g(x)| = 0$. Let $x \in U \backslash Y$. Since $0 \leq 1 - g \leq 1$ we have $|f(x) - f(x)g(x)| \leq |f(x)| < \|f_Y\| + \varepsilon$. Thus $\|f - fg\| < \|f_Y\| + \varepsilon$. Since $fg \in M$ we get $\|f + M\| < \|f_Y\| + \varepsilon$.

Problem 388

Let K be a compact subset of $\mathbb{C}$ with non-empty interior. Let $A(K) = \{f \in C(K) : f \in H(K^0)\}$ where K^0 is the interior of K and $H(K^0)$ is the space of all holomorphic functions on K^0. If $z \in K$ show that $f(z) = \int f d\mu, f \in A(K)$ for some probability measure μ on ∂K.

If $z \in \partial K$ take μ to be δ_z. Suppose $z \in K^0$. Then $|f(z)| \leq \sup\{|f(\zeta)| : \zeta \in \partial(K^0)\}$ by Maximum Modulus Theorem. Let $M = \{f \in C(\partial K) : f \text{ extends to}$ an element of $A(K)\}$. The map $f \in M \to f(z) \in \mathbb{C}$ is a well-defined continuous linear functional since the extension of elements of M is unique. The norm of this functional does not exceed 1. [Let C be the connected component of K^0 containing z. Then $\partial C \subseteq \partial K^0 \subseteq \partial K$ so the values on ∂K determine values on ∂C, hence in C]. By Hahn-Banach Theorem we can extend this to a continuous linear functional on $C(\partial K)$. Hence there exists a complex measure μ on ∂K such that $f(z) = \int f d\mu$ for all $f \in M$. Clearly $\|\mu\| \leq 1$. This fact and the fact $\mu(\partial K) = 1$ together imply that μ is a positive measure (hence a probability measure). [$d\mu = hd\,|\mu|$ with $|h| = 1$ a.e. $[|\mu|]$ and $1 = \int hd\,|\mu|$ so $1 = \int \operatorname{Re} hd\,|\mu| \leq \int 1d\,|\mu| = 1$ so $\operatorname{Re} h = 1 = |h|$ a.e. $[|\mu|]$ which implies $h = 1$ a.e. $[|\mu|]$ and $\mu = |\mu|$].

Problem 389

Let X be a Banach space and $T : X \to X$ be a linear map such that $T^2 = T$ and the null space and range of T are both closed. Show that T is continuous.

Let $M = T^{-1}\{0\}$ and $N = T(X)$. By hypothesis M and N are closed subspaces. Since $x = (x - Tx) + Tx$ we have $X = M + N$. Further $M \cap N = \{0\}$. This implies that the projection maps $M + N \to M$ and $M + N \to N$ have closed graphs and hence are continuous. [Suppose, for example, $\{x_n\} \subseteq M, \{y_n\} \subseteq N, x_n + y_n \to x + y$ (with $x \in M, y \in N$) and $x_n \to z$. Then $y_n \to x + y - z$. Hence $x + y - z \in N$. But then $x - z \in N$ where as x and $z \in M$ so $x - z \in M \cap N = \{0\}$. Thus $z = x$ which is the projection of $x + y$ on M]. If $x_n \to x$ then the projection of x_n on N is Tx_n and that of x is Tx so $Tx_n \to Tx$.

Problem 390

Give a simple proof of the following fact without using Egoroff's Theorem: if $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p < \infty$ then $\|f_n - f\|_p \to 0$.

Proof due to Novinger: $2^p[|f_n|^p + |f|^p] - |f_n - f|^p \to 2^{p+1}|f|^p$ and $2^p[|f_n|^p + |f|^p] - |f_n - f|^p \geq 0$. Just apply Fatou's Lemma.

Problem 391

Show that c is not isometrically isomorphic to c_0.

We show that the closed unit ball of c has extreme points, but that of c_0 has none. Let $\{a_n\}$ be in the closed unit ball of c_0 and choose N such that $|a_n| < \frac{1}{2}$ for $n \geq N$. Let $x_n = y_n = a_n$ for $n = 1, 2, ..., N-1, x_n = a_n + 2^{-n} (n \geq N)$ and $y_n = a_n - 2^{-n} (n \geq N)$. Then $\{a_n\} = \frac{1}{2}\{x_n\} + \frac{1}{2}\{b_n\}$ and $\{x_n\}, \{y_n\}$ are in the closed unit ball. Thus there are no extreme points in the unit ball of c_0. However $(1, 1, ...)$ is an extreme point in the unit ball of c since $(1, 1, ...) = t\{x_n\} + (1-t)\{y_n\}$ implies $1 = tx_n + (1-t)y_n \leq t + (1-t) = 1$ and $x_n = y_n = 1$ for all n if $0 < t < 1$ and $\{x_n\}, \{y_n\}$ belong to the unit ball.

Problem 392

Find extreme points of the closed unit ball Δ of $L^p(\mu), 1 \leq p < \infty$.

If $1 < p < \infty$ then $f \in \Delta$ is an extreme point of Δ if and only if $\|f\|_p = 1$.

Proof: of course, only if holds. Suppose $\|f\|_p = 1$. Suppose $f = tg + (1-t)h$ with $0 < t < 1$ and $g, h \in \Delta$. It is obvious that $\|g\|_p = 1$ and $\|h\|_p = 1$. Now $1 = \int |f|^p = \int |tg + (1-t)h|^p \leq \int [t\,|g| + (1-t)\,|h|]^p \leq \int [t\,|g|^p + (1-t)\,|h|^p] = t + (1-t) = 1$ where we have used the fact that $a \to a^p$ is convex on $[0, \infty)$. Since $a \to a^p$ is strictly convex it follows that $|g| = |h|$ a.e.. Since $|tg + (1-t)h| = t\,|g| + (1-t)\,|h|$ a.e. we get $g = h$ a.e.

If $p = \infty$ then $f \in \Delta$ is an extreme point if and only if $|f| = 1$ a.e.

Proof: suppose $|f| = 1$ a.e.. If $f = tg + (1-t)h$ with $\|g\|_\infty \leq 1$ and $\|h\|_\infty \leq 1$ then $1 = |f| \leq t\,|g| + (1-t)\,|h| \leq t + (1-t) = 1$ so $|g| = 1 = |h|$ a.e. Since $|tg + (1-t)h| = t\,|g| + (1-t)\,|h|$ a.e. we get $g = h$ a.e.. Now suppose $E = \{x : |f(x)| < 1 - \delta\}$ has positive measure for some $\delta > 0$. We have $f = \frac{1}{2}[\{(f+\delta)I_E + fI_{E^c}\} + \{(f-\delta)I_E + fI_{E^c}\}]$ and the functions $(f+\delta)I_E + fI_{E^c}$ and $(f-\delta)I_E + fI_{E^c}$ belong to Δ. It follows that if f is an extreme point then E has measure 0 for each $\delta > 0$ which means $|f| = 1$ a.e.

If $p = 1$ then $f \in \Delta$ is an extreme point if and only if $f = e^{ia}\frac{I_A}{\mu(A)}$ for some $\mu-$ atom A and some real number a.

Proof: let $\|f\|_1 = 1$. For any measurable set A such that $0 < \int_A |f| < 1$ we have $f = \int_A |f| \frac{fI_A}{\int_A |f|} + \int_{A^c} |f| \frac{fI_{A^c}}{\int_{A^c} |f|}$ which shows that f is not an extreme point. Hence, if f is an extreme point then $\int_A |f| = 0$ or 1 for every measurable set A. Let $A = \{f \neq 0\}$. If this set (of positive measure) has a subset B with $0 < \mu(B) < \mu(A)$ then $\int_B |f| \in (0,1)$, a contradiction. Thus, A is necessarily an atom. This implies that f is almost everywhere constant on A. Hence $f = cI_A$ for some constant c. Since $\|f\|_1 = 1$ we have $|c|\,\mu(A) = 1$. This proves the 'only if 'part. Now suppose $f = e^{ia}\frac{I_A}{\mu(A)}$ for some $\mu-$ atom A and some real number a. Suppose $f = tg + (1-t)h$ with $0 < t < 1$ and $\|g\|_1 = \|h\|_1 = 1$. Since A is an atom, g and h are constants on A. Since $1 = \int |f| \leq t \int |g| + (1-t) \int |h| \leq 1$ and $1 = \int_A |f| \leq t \int_A |g| + (1-t) \int_A |h| \leq 1$ we see that $\int_A |g| = \int |g|, \int_A |h| = \int |h|$ (i.e. $g = h = 0$ on A^c) and the constants g and h are such that g/h is non-negative and since $|g| = |h| = \frac{1}{\mu(A)}$ we get $g = h$ a.e.

Problem 393

Let X be a Banach space and T be a bounded operator on it. Show that $\sum_n \|T^n x\| < \infty$ for all $x \in X$ if and only if there is a positive integer N such that $\|T^N\| < 1$.

We give the proof assuming that X is a complex Banach space. The real case can be hndled by complexification. [See Schechter, Principles of Functional Analysis]. If $\|T^N\| < 1$ then any positive integer n can be written as $Nk+j$ with $0 \leq j < N, k \in \{0, 1, 2, ...\}$. We have $\|T^n x\| = \|T^{Nk+j} x\| \leq \|T^N\|^k \max\{\|T^l\| :$

$0 \leq l < N\}$. As $n \to \infty$, $k \to \infty$ too and since $\left\|T^N\right\| < 1$ it follows that $\sum_n \|T^n x\| < \infty$. Conversely let $\sum_n \|T^n x\| < \infty$ for all x. Let $|\lambda| \geq 1$. If $Tx = \lambda x, x \neq 0$ then $\sum_n |\lambda|^n \|x\| < \infty$ a contradiction. Hence $T - \lambda I$ is one-to-one. To prove that it is onto we define $x_n = \frac{1}{\lambda} T x_{n-1} + \frac{1}{\lambda} u$ for $n \geq 1$ where x_0 and u are fixed vectors in X. Note that $x_n = \frac{1}{\lambda^n} T^n x_0 + \sum_{j=0}^{n-1} \frac{1}{\lambda^{j+1}} u$. The series $\sum_{j=0}^{\infty} \frac{1}{\lambda^{j+1}} u$ converges and $\frac{1}{\lambda^n} T^n x_0 \to 0$ because $\sum_n \|T^n x_0\| < \infty$. Hence $\{x_n\}$ converges. Let $x = \lim x_n$. Then $x = \frac{1}{\lambda} T x + \frac{1}{\lambda} u$ which says $Tx - \lambda x = -u$. Since u is arbitrary we have proved that $T - \lambda u$ is onto. By oopen mapping theorem $T - \lambda I$ is invertible. Thus $|\lambda| \geq 1$ implies $\lambda \notin \sigma(T)$. It follows that $\sigma(T)$ is a compact subset of $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and hence the spectral radius ρ of T is less than 1. By the spectral radius formula we see that $\|T^n\| < 1$ for some n.

Remark: if X is finite dimensional then the two equivalent conditions above are equivalent to the condition $T^n x \to 0$ for each x. To see this let $\{e_j : 1 \leq j \leq N\}$ be a basis. Then $\left\|T^n(\sum_{j=1}^N x_j e_j)\right\| = \left\|\sum_{j=1}^N x_j T^n e_j\right\| \leq \sqrt{\sum_{j=1}^N |x_j|^2} \sqrt{\sum_{j=1}^N \|T^n e_j\|^2}$. Thus $T^n x \to 0$ for each x implies $\|T^n\| \to 0$, which implies that the equivalent conditions above hold. Of course $\sum_n \|T^n x\| < \infty$ for all $x \in X$ implies that $T^n x \to 0$ for each x and the three conditions are all equivalent.

Problem 394

Let X be a Banach space and T a bounded self-adjoint operator with $\|T\| \leq 2$. Show that there exist unitary operators U and V such that $T = U + V$.

The closed sub-algebra of $B(X)$ generated by I and T is a commutative C^* algebra with identity.[This is the closure of the set of all polynomials in T]. Hence it is isometrically * - isomorphic to $C(\Delta)$ for some compact Hausdorff space Δ. It suffices to show that if f is a real valued function in $C(\Delta)$ with $\|f\|_\infty \leq 2$ then there exist $g, h \in C(\Delta)$ such that $f = g + h$ and $1 = |g(x)| = |h(x)|$ for all $x \in \Delta$. We just have to take $\operatorname{Re} g(x) = \operatorname{Re} h(x) = \frac{f(x)}{2}$ and $\operatorname{Im} g(x) = -\operatorname{Im} h(x) = \sqrt{1 - \{\frac{f(x)}{2}\}^2}$ to complete the proof.

Problem 395

Let A be a complex algebra with a multiplicative unit e. Let $a \in \mathbb{C} \backslash \{0\}$. If $ae - xy$ is invertible show that $ae - yx$ is also invertible. Is this true for $a = 0$?.

If x or y is invertible show that $ae-xy$ and $ae-yx$ are simultaneously invertible or non-invertible (for any $a \in \mathbb{C}$)

Let $a \neq 0$ and $c = (ae - xy)^{-1}$. Then $(ae - yx)(ycx) = aycx - yxycx = y(ae - xy)cx = yx$ so $(ae - yx)(e + ycx) = ae - yx + yx = ae$. Similarly $(ycx)(ae - yx) = aycx - ycxyx = y(ae - xy)cx = yx$ and $(e + ycx)(ae - yx) = ae - yx + yx = ae$. It follows that $(e - yx)^{-1} = a^{-1}(e + ycx)$. If $a = 0$ this is false: xy may be invertible without yx being so. For example in $B(l^2)$ let $T\{a_n\} = (0, a_1, a_2, ...)$ and $S\{a_n\} = (a_2, a_3, ...)$. Then $ST = I$ but TS is not surjective. Suppose x and $ae - xy$ are invertible. If $a = 0$ then y is invertible and so is yx. If $a \neq 0$ then the first part can be applied. Similar argument shows that if y and $ae - xy$ are invertible so is $ae - yx$.

Problem 396

Give an example of two non-negative definite matrices A and B (over $\mathbb{C}$) such that AB is not non-negative definite.

Let $A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 12 & -6 \\ 4 & -3 \end{pmatrix}$. The quadratic forms corresponding to A and B are $|2x + y|^2$ and $|2x - y|^2$ respectively. AB is not even self adjoint.

Problem 397

Let $S : l^2 \to l^2$ be defined by $S\{a_1, a_2, ...\} = \{0, a_1, a_2, ...\}$. If $T : l^2 \to l^2$ is a linear map such that $TS = ST$ then T is continuous.

We claim that $(Tx)_j$ (the $j{-}th$ coordinate of Tx) depends only on $x_1, x_2, ..., x_j$. In fact $Tx = x_1Te_1 + x_2Te_2 + ... + x_jTe_j + T(0, 0, ...0, x_{j+1}, x_{j+2}, ...)$. The last term is $TS^j(x_{j+1}, x_{j+2}, ...) = S^jT\ (x_{j+1}, x_{j+2}, ...)$ and hence its first j coordinates are 0. This proves the claim. Note that $(T\{x_1, x_2, ...\})_j = (T\{x_1, x_2, ..., x_j, 0, 0, 0, ...\})_j = \sum_{i=1}^{j} x_i(Te_i)_j$. Continuity of T now follows easily by The Closed Graph Theorem.

Remark: $(Te_i)_j = (Te_1)_{j-i+1}$ for each i, j with $i \leq j$ and $(Te_i)_j = 0$ if $i > j$. To see this just note that $(Te_i)_j = (TS^{i-1}e_1)_j = (S^{i-1}Te_1)_j = (Te_1)_{j-i+1}$ if $i \leq j$ and 0 otherwise. [The identity $(Te_i)_j = (Te_1)_{j-i+1}$ is trivial if $i = 1$ so we have assumed above that $i > 1$]. Thus Te_1 determines T completely. However Te_1 cannot be an arbitrary element of l^2. For characterization of operators that commute with S see Hilbert Space Problem Book by Halmos, for example. If $U\{a_n\} = \{0, \alpha_1a_1, \alpha_2a_2, ...\}$ where $\alpha_n > 0$ for all n and $\{\alpha_n\}$ is bounded then any linear map that commutes with U is continuous, by the same argument.

Problem 398

Let $\phi : C \to \mathbb{C}$ be a continuous function, C being the Cantor ternery set. Show that there exists a Borel measurable map $\xi : \phi(C) \to C$ such that $\phi(\xi(z)) = z$ for all $z \in \phi(C)$.

Remark: this follows from fron general theorems on Borel cross sections (cf. Topology by Kuratowski) but a direct elementary proof can be given.

Define $\xi(z) = \inf\{c \in C : \phi(c) = z\}$. Clearly this is a well-defined map from $\phi(C)$ into C and $\phi(\xi(z)) = z$. To show that ξ is measurable we prove that it is lower semi-continuous. Let $z_n \to z$. Write $\liminf \xi(z_n)$ as $\lim \xi(z_{n_j})$ for some $n_j \uparrow \infty$. Then $\phi(\liminf \xi(z_n)) = \lim \phi(\xi(z_{n_j})) = \lim z_{n_j} = z$. By definition of ξ this implies $\xi(z) \leq \liminf \xi(z_n)$ as required.

Problem 399

If X is a commutative C^* algebra with unit and $xx^* = x^*x$ does it follow that we can write x as $\phi(y)$ for some self adjoint vector y and a continous map $\phi : \sigma(x) \to \mathbb{C}$?

Remark: it is know that the answer is 'yes ' if X is the space of bounded operators on a Hilbert space.

The answer is no: let $X = C(T)$ and x be the identity map : $T \to \mathbb{C}$. Suppose there exists a continuous map $\phi : \sigma(x) \to \mathbb{C}$ such that $x = \phi(\xi)$ where ξ is real valued. Since $z = \phi(\xi(z))$ for all $z \in T$ we see that ξ is a one-to-one continuous map from T into $\mathbb{R}$. There is no such map because the range, which is a closed interval, becomes disconnected when one point is removed from it. [ξ^{-1} is automatically continuous].

Problem 400

Let $f \in L^1(\mathbb{R})$ be an odd function. Show that $\sup\{\left|\int_1^{\Delta} \frac{\hat{f}(t)}{t} dt\right| : 1 < \Delta < \infty\} < \infty$.

Remark: this shows that Fourier transform from $L^1(\mathbb{R})$ into the space $C_0(\mathbb{R})$ of continuous functions that vanish at ∞ is *not* onto. For example, if $g(t) = \frac{1}{\ln(t)}$ if $|t| > 1$ and g is continuous (real valued) on $\mathbb{R}$ then ig cannot be the Fourier transform of an L^1 function. [An integrable function is odd if and only if its Fourier transform if purely imaginary]

We have $\int_1^{\Delta} \frac{\hat{f}(t)}{t} dt = \int_1^{\Delta} \frac{1}{t} \int e^{-itx} f(x) dx dt = \int_1^{\Delta} \frac{-2i}{t} \int_0^{\infty} \sin(ty) f(y) dy dt$ [Using the fact that g is odd]. Hence $\int_1^{\Delta} \frac{\hat{f}(t)}{t} dt = (-2i) \int_0^{\infty} \int_1^{\Delta} \frac{\sin(ty)}{t} dt f(y) dy$. Since

$\int\limits_1^{\Delta} \frac{\sin(ty)}{t} dt = \int\limits_y^{\Delta y} \frac{\sin(s)}{s} ds$ there exists a constant $C \in (0, \infty)$ such that $\left| \int\limits_1^{\Delta} \frac{\sin(ty)}{t} dt \right| \leq C$ for all $y \in (0, \infty)$ and $\Delta \in (1, \infty)$. Hence $\left| \int\limits_1^{\Delta} \frac{\hat{f}(t)}{t} dt \right| \leq 2C \int\limits_0^{\infty} |f(y)| \, dy$.

Problem 401

Show that the operator $S : l^2 \to l^2$ be defined by $S\{a_1, a_2, ...\} = \{0, a_1, a_2, ...\}$ has no square root in $B(l^2)$.

The adjoint W of S is defined by $W\{x_n\} = \{x_2, x_3, ...\}$. If S has a square root so does W. Suppose $T \in B(l^2)$ with $T^2 = W$. Let $M = T^{-1}\{0\}$. Then $M \subseteq W^{-1}\{0\} = [e_1]$, the one-dimensional space spanned by $e_1 = \{1, 0, 0, ...\}$. It follows that either $M = \{0\}$ or it is one-dimensional. In the first case T and $W = T^2$ are one-to-one. This is clearly false ($We_1 = 0$) so M is one-dimensional. But $M \subseteq [e_1]$ so $M = [e_1]$. Since $T^2 = W$ and W is onto, so is T. Let $e_1 = Tz$. Of course, $z \neq 0$. Now $Wz = T^2 z = Te_1 = 0$ which implies $z = ce_1$ for some scalar c. Thus $e_1 = Tz = cTe_1 = 0$ a contradiction.

Problem 402 [Extending a metric]

Show that any metric on a subset can be extended to a metric on the big set.

Let $A \subset B$ and d be a metric on A. Fix A. Define $D(x, y) = d(x, y)$ if x and $y \in A, 1$ if x and y are distinct points of $B|A$, 0 if $x = y \in B \backslash A, 1 + d(u, x)$ if $x \in A$ and $y \in B|A, 1 + d(u, y)$ if $y \in A$ and $x \in B|A$. To prove triangle inequality a number of cases have to be considered, but D is indeed a metric on B which extends d.

Problem 403

Show that if a and b are elements in a Banach algebra with unit e then $ab - ba \neq e$.

Suppose $ab - ba = e$. We prove by induction that $a^n b - ba^n = na^{n-1}$. This is true for $n = 1$. Suppose $a^n b - ba^n = na^{n-1}$ for $n \leq m$. Then $a^{m+1}b - aba^m = ma^m$ and $a^m ba - ba^{m+1} = ma^m$. Adding these we get $a^{m+1}b - aba^m + a^m ba - ba^{m+1} = 2ma^m$. Note that $a^m ba - aba^m = a[a^{m-1}b - ba^{m-1}]a = a[(m-1)a^{m-2}]a = (m-1)a^m$. Hence $a^{m+1}b + (m-1)a^m - ba^{m+1} = 2ma^m$. This gives $a^{m+1}b - ba^{m+1} = (m+1)a^m$. This completes the induction argument. Now $\|a^n b - ba^n\| = n \left\|a^{n-1}\right\|$. Since $\|a^n b - ba^n\| \leq \left\|a^{n-1}\right\| \|ab\| + \|ba\| \left\|a^{n-1}\right\|$ we get $n \left\|a^{n-1}\right\| \leq 2 \|a\| \|b\| \left\|a^{n-1}\right\|$. This implies that $a^{n-1} = 0$ for $n > 2 \|a\| \|b\|$.

However the equation $a^n b - ba^n = na^{n-1}$ shows that $a^{n-1} = 0$ whenever $a^n = 0$ and $n \geq 2$. It follows that $a = 0$ which leads to the contradiction $e = ab - ba = 0$.

Remark: in the case of $B(X)$ where X is finite dimensional there is a onle line proof: ab and ba have the same trace so $\dim(X) = tr(e) \neq 0 = tr(ab - ba)$

Problem 404

Brouwer's Fixed Point Theorem says that any continuous map from the closed unit ball of $\mathbb{R}^n$ (or $\mathbb{C}^n$) has a fixed point. Does this extend to infinite dimensional normed linear spaces?

No. Let Δ be the closed unit ball of l^2 and define $f : \Delta \to \Delta$ by $f(x) = (\sqrt{1 - \|x\|^2}, x_1, x_2, ...)$. If $f(x) = x$ then $x_{n+1} = x_n$ for all n which implies $x_n = 0$ for all n. But then $1 = \sqrt{1 - \|x\|^2} \neq 0 = x_1$.

Problem 405

Does there exist a strictly increasing absolutely continuous function on $[0, 1]$ whose derivative vanishes on a set of positive measure?

Yes. There exists a set $E \subseteq [0, 1]$ such that $0 < m(E \cap I) < m(I)$ for every open interval $I \subseteq [0, 1]$. [Start with a Cantor like set of positive measure; in each of the intervals that you remove construct another Cantor like set of positive measure; repeat this process and take the union of all the Cantor like sets that you have constructed]. Let $f(x) = \int_0^1 I_E(x)dx$. Then f has the required properties.

Problem 406

Let $f : [a, b] \to \mathbb{R}$ be a measurable function. Show that f is approximately continuous almost everywhere in the following sense: there is a null set E such that for each $x \in [a, b] \backslash E$ there exists a set A_x containing x such that the restriction of f to A_x is continuous at x and $\frac{m[(x-\delta, x+\delta) \cap A_x]}{2\delta} \to 1$ as $\delta \to 0$.

Let $\varepsilon > 0$. By Lusin's Theorem there exists a continuous function g such that $m\{y : f(y) \neq g(y)\} < \varepsilon$. Now almost all points of $\{y : f(y) = g(y)\}$ have density 1. Let A be the set of all points of $\{y : f(y) = g(y)\}$ of density 1. If $x \in A$ and $f(x) = g(x)$ then the restriction of f to A is continuous at x (by continuity of g) and $\frac{m[(x-\delta, x+\delta) \cap A]}{2\delta} \to 1$ as $\delta \to 0$. It follows that there is a set whose measure is $< \varepsilon$ such that for almost every point x in the complement the conclusion holds. Hence the set of points at which the conclusion does not hold is contained in set whose measure is $< \varepsilon$. Since ε is arbitrary the proof is complete.

Problem 407 [This is same as Problem 23 but the solution is different]

Let $f : [0,1] \to \mathbb{R}$ be a function such that for every $\varepsilon > 0$ there exists $\delta > 0$ with the property that for any finite number of intervals (a_i, b_i) with $\sum(b_i - a_i) \leq \delta$ we have $\left|\sum\{f(b_i) - f(a_i)\}\right| < \varepsilon$. Show that f is Lipschitz.

Remark: the hypothesis becomes stronger if we replace $\left|\sum\{f(b_i) - f(a_i)\}\right| < \varepsilon$ by $\sum |f(b_i) - f(a_i)| < \varepsilon$. Thus if we omit disjointness of the intervals in the definition of absolute continuity we get a Lipschitz function.

Proof: take $\varepsilon = 1$. Given $a < b$ consider the interval (a, b) repeated N times where $N = [\frac{\delta}{b-a}]$. We get $N\,|f(b) - f(a)| < 1$. If $b - a < \delta/2$ we get $|f(b) - f(a)| < \frac{1}{N} < \frac{1}{\frac{\delta}{b-a} - 1} = \frac{b-a}{\delta - (b-a)} < \frac{2(b-a)}{\delta}$. For arbitrary $a < b$ we can find points $\{t_i\}$ such that $a = t_1 < t_2 < ... < t_k = b$ and $t_{i+1} - t_i < \delta/2$ for each i. We get $|f(b) - f(a)| \leq \sum |f(t_{i+1}) - f(t_i)| < \sum \frac{2(t_{i+1} - t_i)}{\delta} = \frac{2(b-a)}{\delta}$.

Problem 408

Prove or disprove that if two functions from $[0,1]$ to $\mathbb{R}$ map null sets to null sets the so does their sum? What about the product?

Both are false. Let $\phi : C \to C \times C$ be a continuous surjective map. [For example $\sum \frac{a_n}{3^n} \to (\sum \frac{a_{2n-1}}{3^n}, \sum \frac{a_{2n}}{3^n})$ is one such map]. Write ϕ as (f, g) so that f and g map C into C. Extend f and g to continuous functions on $[0,1]$ by making then linear on the intervals removed in the construction of C. Since linear maps map null sets to null sets it is easy to see that f and g do the same. If $x \in [0,2]$ then there exist $x_1, x_2 \in C$ such that $x = x_1 + x_2$. Since ϕ is onto there exists $t \in C$ such that $(f(t), g(t)) = \phi(t) = (x_1, x_2)$. Thus $f(t) + g(t) = x$. We have proved that $f + g$ maps C onto $[0,2]$ so it does not map null sets to null sets. Also, e^f and e^g map null sets to null sets (because e^x is Lipschtz) and $(e^f e^g)(C) = e^{f+g}(C) = e^{(f+g)(C)} = e^{[0,1]} = [1, e]$ so the product $e^f e^g$ does not map null sets to null sets.

Problem 409

Let $f : \mathbb{R} \to \mathbb{R}$ and consider the following properties of f :

a) f has intermediate value property (ivp), i.e. $a < b$ and $f(a) < y < f(b)$ or $f(b) < y < f(a)$ implies there exists $c \in (a, b)$ such that $y = f(c)$

b) $a < b$ implies $f([a, b])$ is an interval

c) f maps intervals to intervals

d) $a < b$ implies $f((a, b))$ is an interval

Are these conditions equivalent?

It is easy to see that a), b) and c) are equivalent and that a) implies d). d) does not imply the other conditions: let $f(x) = 0$ for $x < 0, 1$ for $x = 0$ and

$\sin(\frac{1}{x})$ for $x > 0$. Take $a = -1, b = 0, y = \frac{1}{2}$ to see that a) fails. Using the fact that $0, 1 \in f(0, \varepsilon)$ for any $\varepsilon > 0$ it easy to see that a) holds.

Problem 410

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and E be a meaurable set. Suppose $f'(x)$ exists for each $x \in E$. Show that $f(E)$ is Lebesgue measurable and $m(f(E)) \leq \int\limits_E |f'(x)|\, dx$.

See problem 411 for an important application of this problem.

We use Vitali's Theorem [Ref: p. 177 of Cohn's Measure Theory]. We first prove that $m(f(E)) \leq Mm(E)$ if $|f'(x)| \leq M$ on E. Let $\varepsilon > 0$ and $x \in E$. If $f'(x) > 0$ then there exists $h_x > 0$ such that $f(x) \leq f(x+h) \leq f(x) + (M+\varepsilon)h$ for $0 \leq h \leq h_x$ and $|f(x+h) - f(x)| \leq (M+\varepsilon)\,|h|$ for $-h_x \leq h \leq 0$. Similarly if $f'(x) < 0$ then there exists $h_x > 0$ such that $f(x) \leq f(x-h) \leq f(x) + (M+\varepsilon)h$ for $0 \leq h \leq h_x$ and $|f(x+h) - f(x)| \leq (M+\varepsilon)\,|h|$ for $-h_x \leq h \leq 0$. If $f'(x) = 0$ we choose $h_x > 0$ such that $|f(x+h) - f(x)| < \varepsilon/2$ for $|h| \leq h_x$. Let U be an open set such that $E \subseteq U$ and $m(U) < m(E) + \varepsilon$. We may also assume that $(x - h_x, x + h_x) \subseteq U$ for each $x \in E$. Using the numbers h_x it is easy to see that we can cover $f(E)$ by a collection of closed intervals such that for each $f(x)$ in $f(E)$ and each $\eta > 0$ there is an interval in this collection with $f(x)$ as one of the end points whose length is less than η and the other end point belongs to $f(E)$. By Vitali's Theorem there is a countable disjoint sub-collection $I_1, I_2, ...$ such that $f(E) \backslash (I_1 \cup I_2 \cup ...)$ is a null set. This gives $m(f(E)) \leq \sum m(I_n)$. Let the end points of I_j be $f(x_j)$ and $f(y_j)$. We can ensure that $[x_j, y_j] \subseteq U$ for all j, $f([x_j, y_j]) \subseteq I_j$ and $m(I_j) \leq (M+\varepsilon)\,|y_j - x_j|$. Thus $m(f(E)) \leq (M+\varepsilon) \sum |y_j - x_j|$. Since the images of the intervals $[x_j, y_j]$ are disjoint so are these intervals. Hence $m(f(E)) \leq (M+\varepsilon)m(U) < (M+\varepsilon)(m(E)+\varepsilon)$. Letting $\varepsilon \to 0$ we get $m(f(E)) \leq Mm(E)$. We have proved that $m(f(E)) \leq Mm(E)$ if $|f'(x)| \leq M$ on E. By decomposing E into the sets $E \cap \{t_{j-1} \leq |f'(x)| < t_j\}$ where $t_j's$ form a partition of $[0, \infty)$ we see that $m(f(E)) \leq \int\limits_E |f'(x)|\, dx$. Measurabilty of $f(E)$ is easy since f maps null sets to null sets.

Problem 411

A function $f : [0, 1] \to \mathbb{R}$ is absolutely continuous if and only if it is a continuous function of bounded variation and maps null sets to null sets.

If f is absolutely continuous then it is a continuous function of bounded variation and maps null sets to null sets. To prove that f maps null sets to null sets we first prove that given $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite disjoint collection of open intervals $(a_1, b_1), (a_2, b_2), ..., (a_k, b_k)$

with $\sum(b_j - a_j) < \delta$ we have $\sum \sup\{|f(t) - f(s)| : s, t \in [a_j, b_j]\} < \varepsilon$. Indeed $\sup\{|f(t) - f(s)| : s, t \in [a_j, b_j]\} = |f(v_j) - f(u_j)|$ for some $u_j, v_j \in [a_j, b_j]$. Note that $|v_j - u_j| \leq b_j - a_j$ we have $\sum |v_j - u_j| < \delta$. The intervals (u_j, v_j) or (v_j, u_j) are contained in (a_j, b_j) and hence they are disjoint. Hence, if $\delta = \delta(\varepsilon)$ is chosen as in the definition of absolute continuity we get $\sum \sup\{|f(t) - f(s)| : s, t \in [a_j, b_j]\} = \sum |f(v_j) - f(u_j)| < \varepsilon$. Now let $m(E) = 0$ and choose disjoint open intervals $(a_1, b_1), (a_2, b_2), ...$ such that $E \subseteq \bigcup (a_j, b_j)$ and $\sum(b_j - a_j) < \delta$. Then $f(E) \subseteq \bigcup f((a_j, b_j))$ and $\sum m(f((a_j, b_j)) = \sum\{\max f([a_j, b_j] - \min f([a_j, b_j]\} \leq \sum \sup\{|f(t) - f(s)| : s, t \in [a_j, b_j]\} < \varepsilon$. This completes the 'only if' part. Now suppose f is a continuous function of bounded variation and maps null sets to null sets. Since functions of bounded variation are differentiable a.e. there is a null set A such that f is differentiable at each point of E if $S \subseteq A^c$. By previous problem $m(f(E)) \leq \int\limits_E |f'(x)|\, dx$.

Now $|f(b) - f(a)| \leq m(f([a, b]) = m(f(A^c \cap ([a, b])) \leq \int\limits_{A^c \cap ([a,b])} |f'(x)|\, dx \leq \int\limits_{([a,b])} |f'(x)|\, dx$ which clearly implies absolute continuity of f. [In the equality above we have used that fact $m(f(A \cap ([a, b])) \leq m(f(A)) = 0]$.

Problem 412

Let $f : [0, 1] \to \mathbb{R}$ be absolutely continuous with $f'(x) > 0$ a.e.. Then f^{-1} is absolutely continuous on $[f(0), f(1)]$.

We first show that if $f : [0, 1] \to \mathbb{R}$, $E = \{x : f'(x)$ exists and is non-zero$\}$ and if $m(A) = 0$ then $m(f^{-1}(A) \cap E) = 0$.

A corollary of this is the following:

Let $f : [0, 1] \to \mathbb{R}$ and $E \subseteq \{x : f'(x)$ exists and is non-zero$\}$. If $f(E)$ is a null set then so is E.

(Proof just take $A = f(E)$).

Let $F = \{x : f'(x)$ exists and $f'(x) > 1\}$. For any rational number r we define $F_r = \{x \in F : \frac{f(y)-f(x)}{y-x} > 1$ for all $y \in [r, x)\}$. Let B be the set of points of F_0 at which F_0 has density 1. [$m(F_0 \backslash B) = 0$ so B is dense in F_0]. Claim: f is increasing on B. Let $x_1 < x_2$ with $x_1, x_2 \in B$. Since $x_1 \in [0, x_2)$ and $x_2 \in F_0$ we have $\frac{f(x_1)-f(x_2)}{x_1-x_2} > 1$. Hence $f(x_1) < f(x_2)$. This proves the claim. Let $\varepsilon > 0$ and U be an open set containing A such that $m(U) < \varepsilon$. If $x \in B \cap f^{-1}(A)$ then there exists a sequence of intervals $U_{x,n} = (x - r_n, x + r_n)$ such that $r_n \to 0$ ($r_n's$ depend on x), $x \pm r_n \in B, (f(x - r_n), f(x + r_n)) \subseteq U$ and $f(x + r_n) - f(x - r_n) > 2r_n$. [Since $x \in B$ we have $f(x) - f(x - r_n) > r_n$ and since $x + r_n \in B$ we have $f(x + r_n) - f(x) > r_n$]. The intervals $\{U_{x,n} : x \in$

$B\cap f^{-1}(A), n\geq 1\}$ form a Vitali cover of $B\cap f^{-1}(A)$. Hence there is a disjoint sequence $\{(x-r_n, x+r_n)\}$ from the collection such that $m(B\cap f^{-1}(A))\leq \sum 2r_n$. But $(f(x_n-r_n), f(x_n+r_n))\subseteq U$ and the intervals $(f(x_n-r_n), f(x_n+r_n))$ are also disjoint so $m(B\cap f^{-1}(A))<\varepsilon$. This proves that $m(F_0\cap f^{-1}(A))=0$. Similarly $m(F_r\cap f^{-1}(A))=0$ for each r which implies $m(F\cap f^{-1}(A))=0$. Also the inequlaity $f'(x)>1$ can be replaced by $f'(x)>1/n$ and also by $f'(x)<-1/n$ hence $m(E\cap f^{-1}(A))=0$.

Next we prove the following:

Let $f:[0,1]\to\mathbb{R}$ and $E\subseteq\{x: f'(x) \text{ exists}\}$. Then $f(E)$ is a null set if and only if $f'=0$ a.e. on E.

Proof: The inequality $m(f(E))\leq\int_E f'(x)dx$ (see Problem 410) proves that 'if 'part. If $f(E)$ is a null set then above result implies that $f'=0$ a.e. on E.

Now note that $f(x)=f(0)+\int\limits_0^x f'(t)dt$ so f is strictly increasing. Hence f^{-1} exists. Now f^{-1} is continuous and strictly increasing. If $E\subseteq\{x: f'(x)>0\}$ is measurable then $(f^{-1})'$ exists on E. If $m(E)>0$ the it is not true that $f'=0$ a.e. on E and hence $m(f(E))>0$ by above corollary. Thus $m(f(E))=0$ implies $m(E)=0$. Since f is a homeomorphism this is equivalent to the statement $m(E)=0\Rightarrow m(f^{-1}(E))=0$. But any continuous function of bouded variation which maps null sets to null sets is absolutely continous so f^{-1} is absolutely continuous.

Problem 413

Let F be a continous singular probability distribution function. Show that there is a set E of measure 0 such that $F(E)$ has positive measure.

By Problem 410 $m(F(\{x: F'(x)=0\}))=0$. Let $S=\{x: F'(x)=0\}$. Then $m(F(S))=0$ and hence $m(F(S^c))>0$. Take $E=S^c$.

Problem 414

Give an example to show that composition of two absolutely continuous functions need not be absolutely continuous.

Let $f(x)=x^2\sin^2(\frac{\pi}{2x}), x\neq 0, f(0)=0$ and $g(x)=\sqrt{x}$. Then f and g are both absolutely continuous on $[0,1]$. f is absolutely continuous because $|f'(x)|\leq 2+\pi$ for all x. g is so because $\frac{1}{2\sqrt{x}}\in L^1([0,1])$ and $\int\limits_0^x \frac{1}{2\sqrt{y}}dy=g(x)$ for all x. We prove that $g\circ f$ is not of bounded variation on $[0,1]$. This implies that $g\circ f$ is not absolutely continuous. Consider $\sum\limits_{n=1}^{\infty}\left|g(f(\frac{1}{n}))-g(f(\frac{1}{n+1}))\right|=$

$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sin(\frac{n\pi}{2}) - \frac{1}{n+1} \sin(\frac{(n+1)\pi}{2}) \right| = \infty$ because $\left| \frac{1}{n} \sin(\frac{n\pi}{2}) - \frac{1}{n+1} \sin(\frac{(n+1)\pi}{2}) \right| = \frac{1}{n}$ if n is odd and $\frac{1}{n+1}$ if n is even.

Problem 415

Let $f \in L^1([0,1])$. Given $\varepsilon > 0$ show that there is a continuous function g such that $g'(x)$ exists and equals $f(x)$ almost everywhere, $|g(x)| < \varepsilon$ for all x and $g(0) = g(1) = 0$.

Let $\phi(x) = \int\limits_0^x f(t)dt$. There is a partition $\{t_i\}_{1 \leq i \leq k}$ of $[0,1]$ such that the oscillation of ϕ on $[t_{i-1}, t_i]$ is less than ε for each i. There exists a continuous singular function h_i on $[t_{i-1}, t_i]$ such that $h_i(t_{i-1}) = \phi(t_{i-1})$ and $h_i(t_i) = \phi(t_i)$. [If $\phi(t_{i-1}) \neq \phi(t_i)$ we can take any continous singular function on ξ on $[t_{i-1}, t_i]$ and take $h_i = \alpha\xi + \beta$ for suitable α and β. Otherwise we can take $h_i(x) = c + \xi((x - t_{i-1})(t_i - x))$ where ξ is a continuous singular function on $[0, (\frac{t_i + t_{i-1}}{2})^2]$ and $c = h(t_i) - \xi(0)$]. Clealry we can 'patch up ' $h_i's$ into a single continuous singular function h. Let $g = \phi - h$. Then $g' = f$ a.e. and $g(0) = \phi(0) - h(0) = 0, g(1) = \phi(1) - h(1) = 0$. Now let $t_{i-1} \leq x \leq t_i$. Then $|g(x)| = |\phi(x) - h(x)| \leq osc(\phi; [t_{i-1}, t_i]) < \varepsilon$.

Problem 416

Prove or disprove that any function $f : [0,1] \to \mathbb{R}$ is the derivative of some function.

We prove that derivatives have IVP (Intermediate Value Property). Let $\phi : [0,1] \to \mathbb{R}$ be any continuous function and let $E = \{\frac{f(a)-f(b)}{a-b} : a, b \in [0,1], a \neq b\}$. Claim: E is an interval. For any two points $\frac{f(a)-f(b)}{a-b}$ and $\frac{f(c)-f(d)}{d-c}$ in E consider $\gamma : [0,1] \to E$ defined by $\gamma(t) = \frac{f((1-t)a+tc) - f((1-t)b+td)}{(1-t)a+tc-\{(1-t)b+td\}}$. Assuming, without any loss of generality, that $a < b$ and $c < d$ we have $(1-t)a + tc < (1-t)b + td$ for all t so γ is continuous. Since $\gamma(0) = \frac{f(a)-f(b)}{a-b}$ and $\gamma(1) = \frac{f(c)-f(d)}{d-c}$ it follows that any number between $\frac{f(a)-f(b)}{a-b}$ and $\frac{f(c)-f(d)}{d-c}$ is $\gamma(t)$ for some t, hence belongs to E. Thus E is an interval. Now, if f is differentiable on $[0,1]$ then $F = \{f'(x) : 0 \leq x \leq 1\} \subseteq \bar{E}$. By Mean Value Theorem $E \subseteq F$. Since E is an interval so is any set between E and its closure $\bar{E}$. It follows that F is an interval.

Remarks: can we characterize derivatives? Apparently not in any decent way, according to Logicians. A function has SIVP if the image if any open interval is the entire real line. A nowhere continuous function with SIVP exists.

Such a function was constructed by Lebesgue. Since derivatives are necessarily continuous on dense sets (by Baire Category Theorem) the converse the statemenet of this problem is not true.

Problem 417

Prove that a Borel probability measure P on $\mathbb{R}$ is absolutely continuous w.r.t. Lebesgue measure if and only if $\sup\{|P(A-th)-P(A)| : A \text{ Borel}\} \to 0$ as $t \to 0$ for every real number h.

If $P << m$ and $\frac{dP}{dm} = f$ then $|P(A-th)-P(A)| = \left| \int\limits_{A-th} f - \int\limits_{A} f \right| \leq \int\limits_{A} |f(x-th)-f(x)| \leq \int |f(x-th)-f(x)|$ which $\to 0$ by continuity of translates in L^1. Now suppose $\sup\{|P(A-th)-P(A)| : A \text{ Borel}\} \to 0$ as $t \to 0$ for every real number h. Let $\mu_j(A) = \int\limits_{A} \frac{j}{\sqrt{2\pi}} e^{-x^2j^2/2}dx$. We have $\left|(\mu_j * P)(A) - P(A)\right| = \left|\int \mu_j(A-x)dP(x) - P(A)\right|$

$= \left| \int \int\limits_{A-x} \frac{j}{\sqrt{2\pi}} e^{-y^2j^2/2}dydP(x) - P(A) \right| = \left| \int P(A-y) \frac{j}{\sqrt{2\pi}} e^{-y^2j^2/2}dy - P(A) \right| \leq \int |P(A-y)-P(A)| \frac{j}{\sqrt{2\pi}} e^{-y^2j^2/2}dy$

$= \int \left|P(A-\frac{x}{j})-P(A)\right| \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx \leq \int \sup\{\left|P(A-\frac{x}{j})-P(A)\right| : A \text{ Borel}\} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx$ which $\to 0$ as $j \to \infty$ by Dominated Convergence Theorem. It suffuces, therefore, to show that $\mu_j * P << m$ for each j. Now $(\mu_j * P)(A) = \int \int\limits_{A-x} \frac{j}{\sqrt{2\pi}} e^{-y^2j^2/2}dydP(x)$ and $\int\limits_{A-x} \frac{j}{\sqrt{2\pi}} e^{-y^2j^2/2}dy = \int\limits_{A} \frac{j}{\sqrt{2\pi}} e^{-(y-x)^2j^2/2}dy = 0$ for each x if $m(A) = 0$ and hence $(\mu_j * P)(A) = 0$. We have assumed that $\sup\{|P(A-x)-P(A)| : A \text{ Borel}\}$ is a measurable function of x. It is, in fact, a (uniformly) continuous function if $\sup\{|P(A-th)-P(A)| : A \text{ Borel}\} \to 0$ as $t \to 0$ for every real number h. Indeed $\sup\{|P(A-x)-P(A-y)| : A \text{ Borel}\} = \sup\{|P(A-(x-y))-P(A)| : A \text{ Borel}\} \to 0$ as $x-y \to 0$.

Problem 418

Show that $\mathbb{R}^2$ cannot be expressed as disjoint union of circles:

Remark: it is known that $\mathbb{R}^3$ is a disjoint union of circles. Ref. Set Theory For The Working Mathematician by Ciesielski. See also Problem 419 below.

Lemma

Let $C(x,r)$ and $C(y,\rho)$ be disjoint circles with $y \in C(x,r)$. Then $r < \rho/2$.

Proof: we think of $\mathbb{R}^2$ as $\mathbb{C}$. The point $y + \rho e^{i\alpha}$ belongs to $C(x,r) \cap C(y,\rho)$ if $\left|y - x + \rho e^{i\alpha}\right| = r$ which is equivalent to $r^2 + \rho^2 + 2\rho \operatorname{Re}\{(y-x)e^{-\alpha}\} = r^2$ or $\operatorname{Re}\{(y-x)e^{-\alpha}\} = -\rho/2$. If $z = \frac{y-x}{|y-x|}$ then this condition becomes $Re\{ze^{-\alpha}\} = -\rho/2r$. Such an α exists if an only if $|\rho/2r| \leq 1$. Hence, if $C(x,r)$ are $C(y,\rho)$ are disjoint then $\rho > 2r$.

Now suppose $\mathbb{R}^2$ is a disjoint union of circles $\{C^i\}_{i \in I}$. Let $C_1 = C(x_1, r_1)$ be some member of $\{C^i\}_{i \in I}$. Having chosen $C_k = C(x_k, r_k), 1 \leq k \leq m$ we take $C_{m+1} = C(x_{m+1}, r_{m+1})$ to be any circle from $\{C^i\}_{i \in I}$ which contains x_m. This defines a sequence of circles $\{C_n\}$. By the lemma we have $r_{m+1} < r_m/2$. In particular $C_n's$ are disitinct, hence disjoint. We have $|x_n - x_{n+1}| = r_{n+1}$ for all n. Since $r_n < r_1/2^{n-1}$ it follows that $|x_n - x_{n+k}| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + ... + |x_{n+k-1} - x_{n+k}| < r_1/2^n + r_1/2^{n+1} + ... + r_1/2^{n+k-1} = r_1/2^{n-1}$. Hence $\{x_n\}$ is Cauchy. Let $x_0 = \lim x_n$. Let $C = C(x,r)$ be a member of $\{C^i\}_{i \in I}$ which contains x_0. Since $|x_n - x_{n+k}| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + ... + |x_{n+k-1} - x_{n+k}| < r_n/2 + r_n/2^2 + ... + r_n/2^k < r_n$ it follows that $|x_n - x_0| \leq r_n$. This implies $x_0 \notin C(x_{n-1}, r_{n-1})$ because $|x_{n-1} - x_0| \leq |x_n - x_{n-1}| + |x_n - x_0| \leq r_n + r_n < r_{n-1}$. Since $x_0 \in C(x,r)$ it follows that $C(x,r)$ is distinct from each C_n. Thus $C \cap C_n = \emptyset$ for each n. We arrive at a contradiction by showing that if n is so large that $|x_n - x_0| \leq r_n < r/2$ then $C \cap C_n \neq \emptyset$. If $|x_n - x_0| = r_n$ then $x_0 \in C \cap C_n$. Suppose $|x_n - x_0| < r_n$. If we show that $C_n \subseteq B(x,r)$ it would follow, by convexity, that $B(x_n, r_n) \subseteq B(x,r)$ which implies that $x_0 \in B(x_n, r_n) \subseteq B(x,r)$ which contradicts the fact that $|x - x_0| = r$. If it is not true that $C_n \subseteq B(x,r)$ then, since C_n does not intersect the boundary of $B(x,r)$ either we get $C_n \subseteq \{z : |z - x| > r\}$. (By connectedness of C_n). In particular $|x_n - x| > r$. Now consider the continuous function $t \to |(1-t)x + tx_0 - x_n|$. At $t = 0$ the value is $|x - x_n|$. At $t = 1$ its value is $|x_0 - x_n| < r_n$. If we can show that $|x - x_n| > r_n$ we can conclude that there exists $t \in (0,1)$ such that $|(1-t)x + tx_0 - x_n| = r_n$. It follows that $(1-t)x + tx_0 \in C_n$. However $|(1-t)x + tx_0 - x| = t\,|x - x_0| = tr < r$ contradicting the fact that $C_n \subseteq \{z : |z - x| > r\}$. It remains only to show that $|x - x_n| > r_n$. We have $|x - x_n| \geq |x - x_0| - |x_0 - x_n| > r - r_n > r_n$

Problem 419

Can $\mathbb{R}^2$ be expressed as a disjoint union of open balls? What about closed balls of (positive radius)?

Connectedness shows that we cannot express $\mathbb{R}^2$ as a disjoint union of open balls. Suppose $\mathbb{R}^2$ is a disjoint union of closed balls. The closed balls contain points with rational coordinates, so the the collection of these balls is necessarily countable. Consider the intersection of these balls with the unit circle T. Pull back these closed segments by the map $x \to e^{2\pi i x}$ to see that $[0,1]$ is a countable disjoint union of closed sets. Now apply Problem 229 above to get a contradiction.

Problem 420

Prove that there exists a set $A \subseteq \mathbb{R}^2$ such that for each x there is a unique y with $(x, y) \in A$ and for every y the section $\{x : (x, y) \in A\}$ is dense in $\mathbb{R}$. Prove also that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $f((a, b)) = \mathbb{R}$ whenever $a < b$ and f is not continuous at any point.

Remark: f has intermediate value property (IVP) since $f((a, b))$ includes all numbers between $f(a)$ and $f(b)$. A function f such that $f((a, b)) = \mathbb{R}$ whenever $a < b$ is said to have strong IVP or strong Darboux property.

The second part follows immediately from the first: define $f(x)$ by the property $(x, f(x)) \in A$. [For any y the set $f^{-1}(\{y\})$ is dense and hence it intersects (a, b) so $f((a, b)) = \mathbb{R}$].

We construct a subset A_0 of $\mathbb{R}^2$ as follows: $A_0 = \{(x_\alpha, y_\alpha) : \alpha < c\}$ where the points $(x_\alpha, y_\alpha)_{\alpha<c}$ are defined by transfinite induction as follows: the family $\mathcal{F}$ consisting of sets of the type $(a, b) \times \{y\}$ with $a, b, y \in \mathbb{R}$ and $a < b$ has cardinality c. We can write this family as $\{T_a : \alpha < c\}$. We pick points (x_α, y_α) as follows: pick any point (x_1, y_1) in T_1; having picked (x_α, y_α) for $\alpha < \beta$ we pick (x_β, y_β) as any point of $T_\beta \backslash \bigcup_{\alpha<\beta} (\{x_\alpha\} \times \mathbb{R})$. This set is not empty because, if we denote the first projection from $\mathbb{R}^2$ to $\mathbb{R}$ by p_1 then the cardinality of $p_1(\bigcup_{\alpha<\beta} (\{x_\alpha\} \times \mathbb{R})$ is atmost that of) is less than c and the cardinality of $p_1(T_\beta)$ equals c. This defines our set $A_0 = \{(x_\alpha, y_\alpha) : \alpha < c\}$. Now we define A as $A_0 \cup \{(x, 0) : A_0 \cap (\{x\} \times \mathbb{R}) = \emptyset\}$. We now verify that A has the desired properties. Let $x \in \mathbb{R}$. If $A_0 \cap (\{x\} \times \mathbb{R}) = \emptyset$ then $(x, 0) \in A$ and $(x, y) \notin A$ if $y \neq 0$. If $A_0 \cap \{\{x\} \times \mathbb{R}) \neq \emptyset$ then there exists y such that $(x, y) \in A_0 \subseteq A$ and y is unique. Hence for each x there is a unique y with $(x, y) \in A$. Now let $y \in \mathbb{R}$. If $a < b$ then $(a, b) \times \{y\} \in \mathcal{F}$ and hence there exists $\alpha < c$ such that $(a, b) \times \{y\} = T_\alpha$. Now $(x_\alpha, y_\alpha) \in T_\alpha$ so $x_\alpha \in (a, b)$ and $y_\alpha = y$. It follows that $x_\alpha \in \{x : (x, y) \in A\} \cap (a, b)$. Hence $\{x : (x, y) \in A\} \cap (a, b)$ is nonempty whenever $a < b$ proving that $\{x : (x, y) \in A\}$ is dense.

Problem 421

Let A and B be disjoint convex sets in a topological vector space X. If $0 \in A^0$ show that there is a non-zero continuous linear functional x^* on X such that $\operatorname{Re} x^*(a) \leq \operatorname{Re} x^*(b)$ for all $a \in A, b \in B$.

Remarks: the condition $0 \in A^0$ can be replaced by the condition that A has an interior point. If A is open the there is a stronger separation result: see Theorem 3.4 a) of Rudin's Functional Analysis.

We first show that there is a linear functional x^* on X and a real number c such that $\operatorname{Re} x^*(a) \leq c \leq \operatorname{Re} x^*(b)$ for all $a \in A, b \in B$ and then show that

$x^*$is necessarily continuous. Fix $y \in B$. Let $C = A - B + y$. Then $0 \in C^0$ and C is convex. Let $p(x) = \inf\{t > 0 : \frac{1}{t}x \in C\}$. Then p is a seminorm. Assume first that X is a real tvs. On the one dimensional space spanned by y consider the map $ty \to tp(y)$. This is a non-zero linear functional and p dominates it: $tp(y) \leq p(ty)$. [This is an equality if $t \geq 0$ and it holds trivially if $t < 0$]. By Hahn Banach Theorem there exists a linear functional x^* on X extending the map $ty \to tp(y)$ such that $x^*(z) \leq p(z)$ for all $z \in X$. Note that $x^*(a-b+y) \leq p(a-b+y) \leq 1$. Now $y \notin C$ and hence $p(y) \geq 1$. [We have used the facts that C is conmvex and $0 \in C$]. Thus $x^*(a-b+y) \leq 1 \leq p(y)$ and so $x^*(a) \leq x^*(b)$ for all $a \in A, b \in B$. In the complex case we define $y^*(z) = x^*(z) - ix^*(iz)$ to get a complex linear functional y^* with$\operatorname{Re} y^*(a) \leq \operatorname{Re} y^*(b)$ for all $a \in A, b \in B$.

We now prove that any linear functional x^* such that $\operatorname{Re} x^*(a) \leq \operatorname{Re} x^*(b)$ for all $a \in A, b \in B$ is necessarily continuous. Since $x^*(z) - ix^*(iz)$ is continuous iff x^* is, we may restrict ourselves to the case of real scalars. Let U be a symmetric neighborhood of 0 such that $U \subseteq A$. Fix $b \in B$. We have $x^*(u) \leq x^*(b)$ for all $u \in U$. Since U is symmetric this gives $|x^*(u)| \leq |x^*(b)|$ for all $u \in U$. This implies $|x^*(v)| < \varepsilon$ for all $v \in \frac{\varepsilon}{2|x^*(b)|}U$ if $x^*(b) \neq 0$ and $x^* \equiv 0$ of $x^*(b) = 0$.

Problem 422

In Problem 421 above can the assumption that A^0 is non-empty be dropped?

No. Let $X = L^2([0,1]), A = \{f \in X : f \text{ is continuous and } f(\frac{1}{2}) = 0\}, B = \{f \in X : f \text{ is continuous and } f(\frac{1}{2}) = 1\}$. Suppose there exists $g \in (L^2)^* = L^2$ such that $0 = \int 0g \leq \int fg$ for all $f \in B$. Since B is dense we get $\int fg \geq 0$ for all $f \in L^2$ which implies $g = 0$ a.e. [Proof of the fact that B is dense: let $f \in L^2, \varepsilon > 0$. There exists h continuous such that $\|f - h\| < \varepsilon/2$. Let $0 < \delta < \frac{\varepsilon^2}{8(2\|h\|_\infty+1)^2}$. There is a continuous function ϕ such that $\phi = h$ on $[0,1]\backslash(\frac{1}{2}-\delta, \frac{1}{2}+\delta), \phi = 1$ on $(\frac{1}{2}-\frac{\delta}{2}, \frac{1}{2}+\frac{\delta}{2})$ and linear in $[\frac{1}{2}+\frac{\delta}{2}, \frac{1}{2}+\delta]$ as well as on $[\frac{1}{2}-\delta, \frac{1}{2}-\frac{\delta}{2}]$. In this case $\|\phi\|_\infty \leq \max\{\|h\|_2, 1\}$. Hence $\|f - \phi\|_2 < \frac{\varepsilon}{2} + \|h - \phi\|_2 \leq \frac{\varepsilon}{2} + (\|h\|_\infty + \|\phi\|_\infty)\sqrt{2\delta} < \varepsilon$].

Problem 423

Let (X, d) be a compact metric space and $C \subseteq X$ be closed. Let $T : X \to X$ satisfy the condition $d(T(x), T(y)) \geq d(x, y)$ for all x, y. If either $T(C) \subseteq C$ or $C \subseteq T(C)$ show that $T(C) = C$.

This is easy from Problem 121 according to which T is necessarily an isometry of X onto itself. If $T(C) \subseteq C$ apply Problem 121 with C in place of X. If $T(C) \subseteq C$ apply the first case to T^{-1}.

Problem 424 [From stackexchange.com]

Let $\{A_n\}$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, P)$. Show that the following are equivalent:

a) $P\{\limsup A_n\} = 1$
b) $\sum P(A \cap A_n) = \infty$ whenever $P(A) > 0$
c) $P(A \cap A_n) > 0$ for infinitely many n whenever $P(A) > 0$.

Remark: this problem show that if $\{A_n\}$ is independent and $\sum P(A_n) = \infty$ then $\sum P(A \cap A_n) = \infty$ whenever $P(A) > 0$. This follows from Borel-Cantelli Lemma. [This is obviuosly false without independence: take $A_n = A^c$ for all n].

a) implies b): suppose $\sum P(A \cap A_n) < \infty$ for some A with $P(A) > 0$. Then $P\{\limsup(A \cap A_n)\} = 0$, By a) this implies $P(A) = 0$ a contradiction.

b) implies c) is obvious.

c) implies a): suppose a) is false. Then there exists n such that $P\{\bigcup_{j=n}^{\infty} A_j\} < 1$. Let $A = \Omega \backslash \bigcup_{j=n}^{\infty} A_j$. Then $P(A) > 0$ and $P(A \cap A_j) = 0$ for all $j \geq n$ so c) is false.

Problem 425

Describe all Hilbert spaces H such that $\{T \in L(H) : T^2 = 0\}$ is dense in the strong operator topology.

In the finite dimensional case $\{T \in L(H) : T^2 = 0\}$ is a proper closed subset of $L(H)$ so it cannot be dense. We claim that it is dense whenever H is infinite dimensional. Consider a basic neighbourhood $N = \{T : \|Tx_i - T_0x_i\| < \varepsilon_i$ for $1 \leq i \leq k\}$ of an operator T_0 in the strong operator topology. We have to show that N intersects $\{T \in L(H) : T^2 = 0\}$. If $x_j \in span\{x_r : r \neq j\}$ then we can find a smaller neghbourhood of T_0 contained in N such that x_j does not appear in that neighbourhood. Repeated use of the argumnet shows that we may suppose that $\{x_1, x_2, ..., x_n\}$ is lineraly independent. Let $\{x_1, x_2, ..., x_n\} \cup \{y_1, y_2, ..., y_n\} \cup \{z_\alpha\}_{\alpha \in I}$ be a Hamel basis for H. We can choose $y_i's$ in such a way that $\|T_0x_i - y_i\| < \varepsilon$. [No open ball can be contained in a finite dimensional suspace of H. So, there exists y_1 in $B(T_0x_1, \varepsilon)$ such that $\{x_1, x_2, ..., x_n, y_1\}$ is linearly independent. Then choose $y_2 \in B(T_0x_2, \varepsilon)$ such that $\{x_1, x_2, ..., x_n, y_1, y_2\}$ is linearly independent, and so on]. If $Tx_i = y_i, 1 \leq i \leq k, Ty_i = Tz_\alpha = 0, 1 \leq i \leq k, \alpha \in I$ then T extends to a bounded operator on H with $T^2 = 0$. Further $T \in N$.

Problem 426

Let (X, d) be a compact metric space and $x_0 \in X$. Prove that x_0 is an isolated point iff $\{f \in C(X) : f = 0$ in some neighbourhood of $x_0\}$ is closed in $C(X)$.

Remark: $\{f \in C(X) : f = 0$ in some neighbourhood of $x_0\}$ is an ideal in $C(X)$. It follows easily from this problem that every ideal in $C(X)$ is closed iff X is a finite set.

If x_0 is isolated, $f_n = 0$ in $B(x_0, r_n)$ and $f_n \to f$ (uniformly) then $f(x_0) = 0$ and $\{x_0\}$ is a neighbourhood of x_0. For the converse let $f_n(x) = \phi_n(d(x, x_0))$ where $\phi_n(t) = \begin{cases} t \text{ if } |t| > 2/n \\ 0 \text{ if } |t| < 1/n \\ 2(|t| - \frac{1}{n}) \text{ if } 1/n \le |t| \le 2/n \end{cases}$. Note that $\phi_n(t) \to t$ uniformly on $\mathbb{R}$. Hence $f_n(x) \to d(x, x_0)$ uniformly on X. f_n vanishes on $B(x_0, \frac{1}{n})$ and f does not vanish in any neighbourhood of x_0 since $\{x_0\}$ is not open.

Problem 427

Let T be a bounded operator on a Banach space X such that $T(C)$ is closed whenever C is closed and bounded. Show that the range of T is closed.

Let $M = T^{-1}\{0\}$. Suppose $Tx_n \to y$. We have to show that $y \in T(X)$. May suppose $y \neq 0$. Let $\alpha_n = d(x_n, M)$. May suppose $\alpha_n > 0$ for all n. There exists $y_n \in M$ such that $2\alpha_n > \|x_n - y_n\|$. Let C be the closure of $\{\frac{x_n - y_n}{\|x_n - y_n\|} : n \ge 1\}$. Then $T(C)$ is closed. If $\|x_n - y_n\| \to \infty$ then $0 \in T(C)$ (because $0 = \lim T(\frac{x_n - y_n}{\|x_n - y_n\|})$ and $T(\frac{x_n - y_n}{\|x_n - y_n\|}) \in C$ for each n). Let $Tz = 0$ with $z \in C$. Now $u_n = y_n + \|x_n - y_n\| z \in M$ and $\|x_n - u_n\| = \left\|\frac{x_n - y_n}{\|x_n - y_n\|} - z\right\| \|x_n - y_n\|$. By the definition of C we can choose n such that $\left\|\frac{x_n - y_n}{\|x_n - y_n\|} - z\right\| < \frac{1}{3}$. We then get $\|x_n - u_n\| < \frac{1}{3}(2\alpha_n) < \alpha_n$ which contradicts the definition of α_n.

Problem 428

Give a simple proof of Tietze's Extension Theorem for metric spaces.

Let A be a closed subset of a metric space X and $f : A \to [0, 1]$ be continuous. Let $F(x) = \frac{1}{d(x,A)} \inf\{\{1 + f(y)\} d(x, y) : y \in A\} - 1$ for $x \in X \backslash A, F(x) = f(x)$ for $x \in A$. Then F is a continuous function from X to $[0, 1]$ which extends f. It is easy to see that $|[\inf\{\{1 + f(y)\} d(x_1, y) : y \in A\} - d(x_1, y)] - [\inf\{\{1 + f(y)\} d(x_2, y) : y \in A\} - d(x_2, y)]| \le 3d(x_1, x_2)$. It follows that F is the ratio of two continuous functions on A^c. Hence F is continuous on A^c. Now let $x \in A, \{x_n\} \subseteq A^c$ and $x_n \to x$. We have to show that $\frac{1}{d(x_n,A)} \inf\{\{1 + f(y)\} d(x_n, y) : y \in A\} \to f(x) + 1$. Suppose $\liminf \frac{1}{d(x_n,A)} \inf\{\{1 + f(y)\} d(x_n, y) : y \in A\} < f(x) + 1$. Then there exists $\delta > 0$ such that $\frac{1}{d(x_n,A)} \inf\{\{1 + f(y)\} d(x_n, y) : y \in A\} \to f(x) + 1 < f(x) + 1 - \delta$ for infinitely many n. Hence, for such n there exists $y_n \in A$ such that $\{1 + f(y_n)\} d(x_n, y_n) < [f(x) + 1 - \delta] d(x_n, A)$. Since the right side $\to 0$ we see that $y_n \to x$. We have $\{1 + f(y_n)\} d(x_n, y_n) < [f(x) + 1 - \delta] d(x_n, y_n)$. Hence $f(y_n) \to f(x)$ leading to the contradiction $1 + f(x) \le 1 + f(x) - \delta$. Hence $\liminf \frac{1}{d(x_n,A)} \inf\{\{1 + f(y)\} d(x_n, y) : y \in A\} \ge f(x) + 1$. Finally note that if $\varepsilon_n \downarrow 0$ and $(1 + \varepsilon_n) d(x_n, A) > d(x_n, y_n)$ with $y_n \in A$ then $\frac{1}{d(x_n,A)} \inf\{\{1 + f(y)\} d(x_n, y) : y \in A\} \le (1 + f(y_n))(1 + \varepsilon_n)$. Since $d(x_n, A) \to 0$ we get $d(x_n, y_n) \to 0$ and so $y_n \to x$. Thus $\frac{1}{d(x_n,A)} \inf\{\{1 + f(y)\} d(x_n, y) : y \in A\} \le (1 + f(y_n))(1 + \varepsilon_n) \to 1 + f(x)$ and the proof is complete.

Problem 429

For a subset A of a topological space X show that the following are equivalent:

a) ∂A is nowhere dense

b) $A = U \cup B$ with U open and B nowhere dense

c) $A = C \backslash B$ with C closed and B nowhere dense

Remark: $\partial\mathbb{Q}$ has lots of interior points, but the boundary of A has no interior if A is either open or closed.

This problem characterizes sets whose boundaries have no interior: such sets have to diiffer from open/closed by a nowhere dense set.

a) implies b): take $U = A^0$ and $B = A \backslash A^0$.

b) implies c): take $C = \bar{A}$, $B_1 = \bar{A} \backslash A$. Claim: $\bar{A} \backslash A \subseteq \bar{B} \cup \partial U$ (where B is as in b)). Let $x \in \bar{A} \backslash A$. If $x \in \bar{U}$ then, since $x \notin A$ we have $x \notin U$ so $x \in \partial U$. If, on the other hand, $x \notin \bar{U}$ then $x \in \bar{A} | \bar{U} \subseteq \bar{B}$. We have proved the claim. It is trivial to check that ∂U is nowhere dense for any open set U. Also the closure of any nowhere dense set is nowhere dense.

c) implies a): let C, B be as in c). Without loss of generality assume that $B \subseteq C$. Claim: $\partial A \subseteq \bar{B} \cup \partial C$. If $x \in \partial A$ and $x \in (A \cup B)^0$ then there is an open set V such that $x \in V \subseteq A \cup B$. If $x \notin \bar{B}$ then we can conclude that $x \in V \backslash \bar{B} \subseteq V \backslash B \subseteq A$ a contradiction since $V \backslash \bar{B}$ is open (and $x \notin A^0$). Thus $x \in \partial A$ and $x \in (A \cup B)^0$ implies $x \in \bar{B}$. But $A \cup B = C$ so $x \in \partial A$ and $x \notin \bar{B}$ imply $x \notin C^0$ and hence $x \in \partial C$. This proves the claim. Since $\bar{B}$ and ∂C are nowhere dense we are done.

Problem 430

Construct a homeomorphism between two intervals which maps a null set to set of positive measure.

Define $f : C \to [0,1]$ by $f(\sum\limits_{n=1}^{\infty} \frac{2}{3^n}) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}$. Then f has the same value at the end points of any of the intervals removed in the construction of C. By making f constant in the intervals removed we get an increasing continuous function f from $[0,1]$ onto itself. Let $g(x) = x + f(x), 0 \leq x \leq 1$. Clearly, g is continuous and strictly increasing. Hence it is a homeomorphism from $[0,1]$ onto $[0,2]$. We claim that $m(g(C)) = 1$. If $(a_n, b_n), n = 1, 2, ...$ are the intervals removed in the construction of C then $g([a_n, b_n]) = c_n + [a_n, b_n]$ where $c_n = f(a_n) = f(b_n)$., Hence $m(g([0,1] \backslash C)) = \sum\limits_{n=1}^{\infty} m(c_n + [a_n, b_n]) = m([0,1] \backslash C) = 1$ so $m(g(C)) = 2 - 1 = 1$. [$\frac{g(x)}{2}$ is a homeomorphism of $[0,1]$ onto itself which maps C onto a set of measure $\frac{1}{2}$].

Problem 431

Let $f, f_n(n = 1, 2, ...) \in L^1(\mu)$ where μ is a finite positive measure. If $\int\limits_E f_n d\mu \to \int\limits_E f d\mu$ for every measurable set E show that $\liminf f_n \leq f \leq \limsup f_n$ a.e. provided $\inf\{f_k : k \geq n\} \in L^1(\mu)$ for each n.

Let $g_n = \inf\{f_k : k \geq n\}$. By Fatou's Lemma we have, for any E, $\int_E \liminf\limits_k (f_k - g_n) d\mu \leq \liminf\limits_k \int_E (f_k - g_n) d\mu$ for each n. This gives $\int_E \liminf\limits_k f_k d\mu \leq \liminf\limits_k \int_E f_k d\mu = \int_E f$. Take $E = \{f < \liminf\limits_k f_k\}$ to get $\mu(E) = 0$. This gives $\liminf f_n \leq f$ a.e.. To get the second inequality change f_n to $-f_n$ and f to $-f$.

Problem 432

Let $P, P_n(n = 1, 2, ...)$ be probability measures on a metric space (X, d) such that $P_n(C) \to P(C)$ for every closed set C. Show that $P_n(E) \to P(E)$ for every Borel set E.

Let E be a Borel set and $\varepsilon > 0$. There exists a closed set C and an open set U such that $C \subseteq E \subseteq U$ and $P(U \backslash C) < \varepsilon$. There exists an integer k such that $|P_n(U) - P(U)| < \varepsilon$ and $|P_n(C) - P(C)| < \varepsilon$ for $n \geq k$. We have $P_n(E) \leq P_n(U) < P(U) + \varepsilon < P(C) + 2\varepsilon \leq P(E)) + 2\varepsilon$ and ${}_n(E) \geq P_n(C) > P(C) - \varepsilon > P(U) - 2\varepsilon \geq P(E) - 2\varepsilon$ for $n \geq k$.

Problem 433

Let $\int \log(1 + cf) dP \leq 0$ for all $c \in \mathbb{C}$ where P is a probability measure and $f \in L^1(P)$. Show that $f = 0$ a.s.

We have $\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|1 - e^{it}\right| dt = 0$. This implies $\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|1 - ce^{it}\right| dt = 0$ if $|c| = 1$. By Cauchy's Theorem we have $\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|1 - ce^{it}\right| dt = 0$ if $|c| < 1$. If $|c| > 1$ then $\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|1 - ce^{it}\right| dt = \log|c| + \frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|\frac{1}{c} - e^{it}\right| dt = \log|c|$. Thus $\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|1 - fe^{it}\right| dt = (\log|f|)^+$. Hence $\int (\log|f|)^+ dP = \frac{1}{2\pi}\int\limits_0^{2\pi} \int \log\left|1 - fe^{it}\right| dP dt \leq 0$ which implies $(\log|f|)^+ = 0$ a.s. and so $|f| \leq 1$ a.s.. Now replace f by rf where $r > 0$ to get $|rf| \leq 1$ a.s. for each $r > 0$. We get $f = 0$ a.s..

Problem 434 [This and the next few problems are from Dieudonne, Treatise on Analysis, Vol. 2]

Let A be a bounded open convex set in $\mathbb{R}^n$. Show that $m(A-A) \leq \binom{2n}{n} m(A)$.

Let $B = A - A$. Then B is a bounded convex symmetric open set containing 0. Let $\rho(x) = (\inf\{t > 0 : tx \in B^c\})^{-1}$ for $x \in B\backslash[0]$ and $\rho(0) = 0$. Note that $0 \leq \rho(x) < 1, (\rho(x))^{-1}x \in \partial B$ and ρ is continuous on B. [See my notes on convexity (convexity.pdf)]. For $0 < t \leq 1$ let $B_t = \{x \in B : \rho(x) < t\}$. It is easy to see from the definition that $B_1 = B$ and $\rho(tx) = t\rho(x)$ so $B_t = tB$. Claim 1: $m(A \cap (A+x)) \geq (1-\rho(x))^n m(A)$. To show this we note that if $x \in B$ and $0 < \varepsilon < 1 - \rho(x)$ then we can write $\frac{1}{\rho(x)+\varepsilon}x = a_1 - a_2$ with $a_1, a_2 \in A$ and $(1-\rho(x)-\varepsilon)A + (\rho(x)+\varepsilon)a_1 = (1-\rho(x)-\varepsilon)A + (\rho(x)+\varepsilon)a_2 + x$. Since the left side of this equality is contained in A and the right side is contained in $A+x$ we see that $m(A \cap (A+x)) \geq m((1-\rho(x)-\varepsilon)A + (\rho(x)+\varepsilon)a_1) = (1-\rho(x)-\varepsilon)^n m(A)$. Claim 1 follows by letting $\varepsilon \to 0$.

Claim 2: $\int\limits_B (I_A * I_{-A})(x)dx = (m(A))^2$. For this note that $\int\limits_B I_A(x+y)dy = m(B \cap (A-x)) = m(A-x) = m(A)$ if $x \in A$. Hence $\int\limits_A \int\limits_B I_A(x+y)dydx = m^2(A)$. Fubini's Theorem shows that the left side of this equation is $\int\limits_B \int\limits_A I_A(x+y)dydx$ and Claim 2 follows by noting that $\int\limits_A I_A(x+y)dy = \int I_A(x+y)I_A(y)dy = \int I_A(x-y)I_A(-y)dy$

$= \int I_A(x-y)I_{-A}(y)dy = (I_A * I_{-A})(x)$. Claim 3: $m(A) \geq \int\limits_B (1-\rho(x))^n dx$.

We have $m^2(A) = \int\limits_B (I_A * I_{-A})(x)dx$

$= \int\limits_B (I_A * I_{-A})(x)dx = \int\limits_B m(x-A) \cap (-A))dx = \int\limits_B m(-x-A) \cap (-A))dx = \int\limits_B m((x+A) \cap A)dx \geq \{\int\limits_B (1-\rho(x))^n dx\} m(A)$ where we have used the fact that B is symmetric. This proves claim 3. We now compute $\int\limits_B (1-\rho(x))^n dx$ as follows: let $\{t_i : 0 \leq i \leq k\}$ be a partition of $[0,1]$. Then $\int\limits_B (1-\rho(x))^n dx =$

$\sum\limits_{i=0}^{k-1} \int\limits_{\{x\in B:t_i\leq \rho(x)<t_{i+1}\}} (1-\rho(x))^n dx$. Note that $\int\limits_{\{x\in B:t_i\leq \rho(x)<t_{i+1}\}} (1-\rho(x))^n dx$ lies between $(1-t_{i+1})^n m(\{x \in B : t_i \leq \rho(x) < t_{i+1}\})$ and $(1-t_i)^n m(\{x \in B : t_i \leq \rho(x) < t_{i+1}\})$. Since $m(\{x \in B : t_i \leq \rho(x) < t_{i+1}\}) = m(B_{t_{i+1}} - B_{t_i}) = (t_{i+1}^n - t_i^n)m(B)$ (because $B_t = tB$) we see that $\int\limits_{\{x\in B:t_i\leq \rho(x)<t_{i+1}\}} (1-\rho(x))^n dx$ lies between $(1-t_{i+1})^n(t_{i+1}^n - t_i^n)m(B)$ and $(1-t_i)^n(t_{i+1}^n - t_i^n)m(B)$. However (applying Mean Value Theorem to $(t_{i+1}^n - t_i^n)$ we see that) $\sum\limits_{i=0}^{k-1}(1-t_{i+1})^n(t_{i+1}^n - t_i^n)m(B)$ and $\sum\limits_{i=0}^{k-1}(1-t_i)^n(t_{i+1}^n - t_i^n)m(B)$ both converge to $\int\limits_0^1(1-t)^n nt^{n-1}dt m(B)$ as the norm of the partition $\{t_i\}$ tends to 0. It follows that $m(A) \geq m(B)\int\limits_0^1(1-t)^n nt^{n-1}dt$. It remains only to show that $\int\limits_0^1(1-t)^n nt^{n-1}dt = \frac{1}{\binom{2n}{n}}$. This is a standard formula in Statistics; See 'beta distribution ' in Bickel and Docksum.

Problem 435

Let T be a bounded operator on a Banach space X such that $\|T^n x\|^{1/n} \to 0$ for each x. Show that the spectral radius $\rho(T)$ of T is 0.

Let c be a non-zero scalar and $T_n x = \frac{1}{c^n}T^n x$. Since $\sup\{\|T_n x\| : n \geq 1\} < \infty$ for each x we can apply Uniform Boundedness Principle to conclude that $\sup\{\|T_n\| : n \geq 1\} < \infty$ which shows $\sup\{\frac{1}{|c^n|}\|T^n\| : n \geq 1\} < \infty$ for each c. This gives $\rho(T) \leq |c|$ for each c.

Problem 436

Let A be a complex Banach algebra and $x, y \in A$ with $\|e^{cx}ye^{-cx}\| \leq M\|y\|$ with M independent of c, x, y. Show that $xy = yx$.

Let $f : \mathbb{C} \to A$ be defined by $f(c) = e^{cx}ye^{-cx}$. If $x^* \in A^*$ then $x^* \circ f$ is an entire function: $\frac{f(c+h)-f(c)}{h} = \frac{e^{(c+h)x}ye^{-(c+h)x} - e^{cx}ye^{-cx}}{h}$
$= \frac{e^{cx}e^{hx}ye^{-cx}e^{-hx} - e^{cx}ye^{-cx}}{h} = e^{cx}\frac{e^{hx}ye^{-hx}-y}{h}e^{-cx} \to e^{cx}(xy - yx)e^{-cx}$ as $h \to 0$. [Estimate the norm of $\frac{e^{cx}e^{hx}ye^{-cx}e^{-hx} - e^{cx}ye^{-cx}}{h} - \frac{e^{cx}(1+hx)ye^{-cx}(1-hx) - e^{cx}ye^{-cx}}{h}$ for a justification]. The hypothesis implies that $x^* \circ f$ is bounded. By Liouville's Theorem we get $(x^* \circ f)(c) = (x^* \circ f)(0)$ for all c and $\frac{d}{dc}(x^* \circ f)(c) = 0$ so $e^{cx}(xy - yx)e^{-cx} = 0$ for all c. Put $c = 0$.

Problem 437

Prove or disprove: the collection of all bounded operators on a Hilbert space H with spectral radius 0 is closed set in $B(H)$ (with the operator norm).

Remark: if H is finite dimensional then any nilpotent operator S on H satisfies $S^N = 0$ where N is the dimension of H. Hence limits of such operators are also nilpotent. Also $\rho(S) = 0$ implies that 0 is the only eigen value which implies that $S^N = 0$.

False. We show that there exists a sequence of nilpotent operators converging to an operator T with $\rho(T) > 0$. Let $H = l^2$ and let $\{e_n\}$ be the standard orthonormal basis. We define numbers $\alpha_1, \alpha_2, ...$ as follows: let $\alpha_n = e^{-m}$ if there exists a positive integer j and a non-negative integer m such that $n = 2^m(2j+1)$. [Since every positive integer has this form and since j, m are unique we have defined α_n for each n]. Let $Te_n = \alpha_n e_{n+1}$ for all n. T defines a bounded operator with $\|T\| \leq \sup\{\alpha_n : n \geq n\}$. We claim that $\rho(T) \geq 1$. It is easy to see that $\left\|T^k\right\| = \sup\{\alpha_n \alpha_{n+1} ... \alpha_{n+k-1} : n \geq 1\}$. In particular $\left\|T^k\right\| \geq \alpha_1 \alpha_2 ... \alpha_k = e^{-\frac{p(p+1)}{2}}$ where $p = [\frac{\log k}{\log 2}]$. It follows that $\liminf \left\|T^k\right\|^{1/k} \geq 1$. We now define $T_r e_n = \alpha_n e_{n+1}$ if n is not of the type $2^r(2j+1)$ and 0 if n has this form. Since at least one of the numbers $n, n+1, ..., n+2^{r+1}-1$ has the form $2^r(2j+1)$ it follows that $T_r^{2^r} = 0$. [One of $n, n+1, ..., n+2^r-1$ is divisible by 2^r. If that number is a then either a has the desired form or it is of the form 2^{r+i} and $2^r(2^i+1)$ has the desired form. This last number differs from a by 2^r is ir must belong to $n, n+1, ..., n+2^{r+1}-1$]. Thus T_r is nilpotent for each r. Also $\|(T-T_r)(e_n)\| = e^{-r}$ if n is of the type $2^r(2j+1)$ and 0 otherwise. It follows that $\|T - T_r\| \leq e^{-r} \to 0$.

Problem 438

Let Ω be a compact metric space and μ be a regular Borel complex measure on it such that $\int fgd\mu = (\int fd\mu)(\int gd\mu)$ for all $f, g \in C(\Omega)$. Show that there exists $x \in \Omega$ such that $\mu = \delta_x$ or $\mu = 0$.

If $f, g \in L^2(|\mu|)$ and $\varepsilon > 0$ then we can find continuous functions f_1 and g_1 such that $\|f - f_1\|_2 < \varepsilon$ and $\|g - g_1\|_2 < \varepsilon$ where $\|\|_2$ is the norm in $L^2(|\mu|)$. It follows that $\|fg - f_1 g_1\|_1 \leq \|f\|_2 \|g - g_1\|_2 + \|g_1\|_2 \|f - f_1\|_2 < \varepsilon(\|f\|_2 + \|g\|_2 + \varepsilon)$. Using this and the fact that $\int f_1 g_1 d\mu = (\int f_1 d\mu)(\int g_1 d\mu)$ it follows easily that $\int fgd\mu = (\int fd\mu)(\int gd\mu)$. Thus $\mu(A \cap B) = \mu(A)\mu(B)$ for all Borel sets A and B. In particular $\mu(A) = \mu^2(A)$ and $\mu(A) = 0$ or 1 for any Borel set A. Let S be the support of μ. If $a, b \in S$ and $a \neq b$ then there exist disjoint open balls U and V containing a and b respectively. Hence $\mu(U)\mu(V) = \mu(U \cap V) = 0$. Thus $\mu(U) = 0$ or $\mu(V) = 0$. This contradicts the definition of S. Hence $S = \{x\}$ for some x. Since $|\mu|(\{x\}^c) = 0$ we get $\mu = c\delta_x$. Clearly $c = 0$ or 1.

Problem 439

Define $T : L^2 \to L^2$ by $Tf(x) = \int e^{-|x-y|} f(y)dy$ [where L^2 stands for (complex) $L^2(\mathbb{R})$]. Show that T is a positive operator whose norm is 2. Also show that $\sigma(T) = [0, 2]$.

We have $e^{-|x|} = \int e^{itx} \frac{1}{\pi(1+t^2)} dt$ so $< Tf, f >= \int \int \int e^{it(x-y)} \frac{1}{\pi(1+t^2)} dt f(y)dy[f(x)]^- dydx$ and for fixed t we have $\int \int e^{it(x-y)} f(y)dy[f(x)]^- dydx = \left|\int e^{-itx} f(x)dx\right|^2 \geq 0$. Hence T is a positive operator. From the inequality $\|\phi * f\|_2 \leq \|\phi\|_1 \|f\|_2$ with $\phi(t) = e^{-|t|}$ we see that $\|Tf\|_2 \leq \int e^{-|t|} dt = 2$ so $\|T\| \leq 2$. If we show that $(0, 2) \subseteq \sigma(T)$ it would follow that $[0, 2] \subset \sigma(T) \subseteq [0, 2]$ so $\sigma(T) = [0, 2]$ which also implies that $\|T\| = 2$. Fix $\lambda \in (0, 2)$ and write λ as $\frac{2}{1+\alpha^2}$ with $\alpha \in (0, \infty)$. Let $f_n(x) = e^{i\alpha x} e^{-|x|/n}$. We claim that $\|Tf_n - \lambda f_n\| \to 0$ proving that $\lambda \in \sigma(T)$ (because $\|f_n\| \to \infty$). [We have dropped the subscript in $\|\|_2$). Now $\|Tf_n - \lambda f_n\| = \left\|\hat{\phi}\hat{f}_n - \lambda \hat{f}_n\right\|$. Let us compute $\hat{f}_n$: $\hat{f}_n(x) = \frac{1}{\sqrt{2\pi}} \int e^{-itx} e^{i\alpha t} e^{-|t|/n} dt = \frac{n}{\sqrt{2\pi}} \int e^{-insx} e^{i\alpha ns} e^{-|s|} ds = n\sqrt{2\pi} \frac{1}{1+n^2(\alpha+x)^2}$. Also $\hat{\phi}(x) = \sqrt{2\pi} \frac{1}{1+x^2}$. It remains to show that $\int \left|\frac{1}{1+x^2} \frac{n}{1+n^2(\alpha+x)^2} - \frac{1}{1+\alpha^2} \frac{n}{1+n^2(\alpha+x)^2}\right|^2 dx \to$ 0 as $n \to \infty$. For $|\alpha + x| \geq 1$ we have $\frac{n}{1+n^2(\alpha+x)^2} \leq \min\{1, \frac{1}{1+(\alpha+x)^2}\}$ and for $0 < |\alpha + x| < 1$ we have $\left|\frac{1}{1+x^2} - \frac{1}{1+\alpha^2}\right| \frac{n}{1+n^2(\alpha+x)^2} \leq \left|\frac{1}{1+x^2} - \frac{1}{1+\alpha^2}\right| \frac{1}{2|\alpha+x|} \leq \frac{|\alpha-x|}{(1+x^2)(1+\alpha^2)}$. Using these and Dominated Convergence Theorem we get $\int \left|\frac{1}{1+x^2} \frac{n}{1+n^2(\alpha+x)^2} - \frac{1}{1+\alpha^2} \frac{n}{1+n^2(\alpha+x)^2}\right|^2 d$ 0.

Problem 440

[This and next few problems are taken from Springer book on Banach Space Theory]

If $\|x + y\| = \|x\| + \|y\|$ then $\|tx + sy\| = t\|x\| + s\|y\|$ for all $t, s \geq 0$.

If $t \geq s$ then $\|tx + sy\| = \|t(x + y) + (s - t)y\| \geq t\|x + y\| - (t - s)\|y\| = t\|x\| + t\|y\| - (t - s)\|y\| = t\|x\| + s\|y\|$ and since the reverse inequality also holds we get $\|tx + sy\| = t\|x\| + s\|y\|$. If $t < s$ replace (t, s) by (s, t) and (x, y) by (y, x).

Problem 441

Prove or disprove: if $\|\|_1$ and $\|\|_2$ are equivalent norms on X then the closed unit balls of $(X, \|\|_1)$ and $(X, \|\|_2)$ are homeomorphic.

True. A homeomorphism is given by $f(x) = \frac{\|x\|_1}{\|x\|_2} x$ if $x \neq 0, f(0) = 0$.

[Thus closed unit balls under any two norms are homeomorphic if X is finite dimensional]

Problem 442

If a linear subspace M of a Banach space X is a G_δ set show that it is necessarily closed. Deduce that if a normed linear space X is homeomorphic to a complete metric space then it is a Banach space.

Suppose $E = \bar{M}\backslash M$ is non-empty. Let $x \in \bar{M}\backslash M$. By hypothesis $M = \bigcap_n U_n$ with each U_n open in X. Let $V_n = \bar{M} \cap U_n$. Then V_n is open in $\bar{M}$. Also $M = \bigcap_n V_n$. Note that $\bar{M} = E \cup M$. If we show that E and M are both of first category in $\bar{M}$ we would get a contradiction (by Baire Category Theorem). First note that $\bar{M}\backslash Vn$ is (closed and) nowhere dense because $M \subseteq V_n$ so V_n is dense in $\bar{M}$. Hence $E = \bigcup_n (\bar{M}\backslash V_n)$ is of first category in $\bar{M}$. Now $x + M \subseteq E$ and hence $x + M$ is also of first category in $\bar{M}$. It follows that M is also of first category in $\bar{M}$. This finishes the proof.

Second part: let Y be the Banach space obtained by completing X. The hypothesis implies that X is a G_δ in Y and the first part shows that X is closed in Y.

[In particular an incomplete normed linear space cannot be homeomorphic to a Banach space].

Problem 443

If X is a normed linear space on which all norms are equivalent then the space is finite dimensional.

Let X be an infinite dimensional normed linear space. There exists a linear functional f on X which is not continuous. Let $x_0 \in X\backslash\{0\}$ and $\|x\|' = \|x\| + \|x_0\| \, |f(x)|$. If this norm is equivalent to the original norm then there exists a finite constant C such that $\|x\| + \|x_0\| \, |f(x)| \leq C \|x\|$ which implies that $|f(x)| \leq \frac{C-1}{\|x_0\|} \|x\|$ contradicting the fact that f is not continuous.

Problem 444

Prove the following generalization of Parallelogram identity:

$$\sum_{\varepsilon_i \in \{-1,1\} \forall i} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^2 = 2^n \sum_{i=1}^{n} \|x_i\|^2.$$ where $x_i's$ are vectors in a Hilbert space.

Proof is by induction on n. For $n = 1$ this is trivial and for $n = 2$ it reduces to the ordinary parallelogram identity. Suppose this holds for $n = k$. Consider $\displaystyle\sum_{\varepsilon_i \in \{-1,1\} \forall i \leq k+1} \left\| \sum_{i=1}^{k+1} \varepsilon_i x_i \right\|^2 = \sum_{\varepsilon_i \in \{-1,1\} \forall i \leq k} 2\left[\left\| \sum_{i=1}^{k} \varepsilon_i x_i \right\|^2 + \|x_{k+1}\|^2\right] =$ $\displaystyle 2^{k+1} \sum_{i=1}^{k} \|x_i\|^2 + 2^{k+1} \|x_{k+1}\|^2 = 2^{k+1} \sum_{i=1}^{k+1} \|x_i\|^2.$

Problem 445

If the normed linear space X finite dimensional then the only dense convex set in X is X itself. If X is infinite dimensional there exist two disjoint dense convex sets whose union is X.

Let C be a dense convex set in a finite dimensional normed linear space X. Fix $x \in X$ and consider $\{t \geq 0 : tx \in C\}$. This is an interval in $[0, \infty)$. If it is bounded interval with right end point s then tx belongs to the exterior (i.e. $X \backslash \bar{C}$) of C which contradicts the hypothesis. [See my notes 'convexity.pdf ']. Hence $tx \in C$ for all $t \geq 0$ for all $x \in X$, so $C = X$. Now let X be an infinite dimensional normed linear space. There exists a linear functional f on X which is not continuous. Let $C_1 = \{x : f(x) < 0\}$ and $C_2 = \{x : f(x) \geq 0\}$. Clearly these are disjoint convex sets. To show that these two sets are both dense it suffices to show that $f^{-1}\{a\}$ is dense for each real number a. The range of f is a subspace of the scalars and it is not $\{0\}$ so f is onto. Let $f(x_0) = a$. Then $f^{-1}\{a\} = x_0 + f^{-1}\{0\}$. It suffices to show that $f^{-1}\{0\}$ is dense. If it is not there exists $x \in X \backslash [f^{-1}\{0\}]^-$. Let $y \in [f^{-1}\{0\}]^- \backslash f^{-1}\{0\}$. y exists because $f^{-1}\{0\}$ is not closed. Now $f(y - ax) = 0$ where $a = \frac{f(y)}{f(x)}$. Hence $y - ax \in f^{-1}\{0\} \subseteq [f^{-1}\{0\}]^-$ and $ax \in [f^{-1}\{0\}]^-$ too (because $y \in [f^{-1}\{0\}]^-$). Since $a \neq 0$ we get $x \in [f^{-1}\{0\}]^-$, a contradiction.

Problem 446

Any real valued Lipschitz function on a subset of a metric space can be extended to a Lipschitz function on the whole space with the same Lipschitz constant.

Suffices to consider the case when the Lipschitz constant is 1. Let $A \subseteq X$ where (X, d) is a given metric space and $f : A \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq d(x, y)$ $\forall x, y \in A$. Let $F(x) = \inf\{f(y) + d(x, y) : y \in A\}$ for $x \in X$. Fix $a \in A$. For any $y \in A$ we have $f(y) + d(x, y) = f(a) + d(x, y) - \{f(a) - f(y)\} \geq f(a) + d(x, y) - d(a, y) \geq f(a) - d(x, a)$. This proves that $F(x) \geq f(a) - d(x, a)$. In particular $F(x) > -\infty$. Of course $F(x) < \infty$ so F is real valued. Also, if $x \in A$ then we can use the inequality established above with $a = x$ to get $F(x) \geq f(a) - d(x, a) = f(x)$. Since $F(x) \leq f(x) + d(x, x)$ by definition we see that F extends f. Now $F(x_1) - F(x_2) - \varepsilon < F(x_1) - \{f(y) + d(x_2, y)\}$ for some $y \in A$ and hence $F(x_1) - F(x_2) - \varepsilon < f(y) + d(x_1, y) - \{f(y) + d(x_2, y)\} \leq d(x_1, x_2)$. Let $\varepsilon \to 0$ to get $F(x_1) - F(x_2) \leq d(x_1, x_2)$. Interchange x_1 and x_2 to get $-\{F(x_1) - F(x_2)\} \leq d(x_1, x_2)$.

Problem 447

Let M be a closed subspace of $C[0, 1]$. If every function in M is continuously differentiable show that M is finite dimensional.

Remark: if a closed subspace of $L^p(\mu)$ is contained in $L^\infty(\mu)$ where μ is a probability measure and $0 < p < \infty$ then the subspace is finite dimensional. See page 111 of Rudin's Functional Analysis for a proof of the result of Grothendieck.

Define $T : M \to C[0,1]$ by $Tf = f'$. T is linear and it has closed graph. Hence there exists an integer N such that $\|f'\|_\infty \leq N \|f\|_\infty$ for all $f \in M$. Claim: $\dim(M) \leq N$. Let $\{x_i : 0 \leq i \leq k\}$ be a partition on $[0,1]$ with $\max\{x_{i+1} - x_i : 0 \leq i < k\} < 1/2N$. Define $S : M \to \mathbb{R}^{k+1}$ by $Sf = (f(x_0), f(x_1), ..., f(x_k))$. If we show that the linear map S is one-to-one we can conclude that $\dim M \leq k+1$. Suppose $Sf = 0$ and $\|f\|_\infty = 1$. If $x \in [0,1]$ then there exists i such that $x \in [x_{i-1}, x_i]$ and Mean Value Theorem gives $|f(x) - f(x_i)| \leq \|f'\|_\infty (x_i - x_{i-1}) < \frac{N}{2N}$. Since $f(x_i) = 0$ this gives $|f(x)| < 1/2$ for each x contradicting the fact that $\|f\|_\infty = 1$.

Problem 448

Let H be a separable Hilbert space with ONB $\{e_1, e_2, ...\}$. Let C be the closed convex hull of $\{e_1, e_2, ...\}$. Show that the interior of C is empty.

Remark: it can be shown that the closed convex hull of any weakly convergent sequence in a Banach space has no interior. [Due to Vesely and Zanco].

Proof: suppose $x \in C^0$. Let $y = \sum_{n=1}^{\infty} \frac{1}{n} e_n$. Then $x + \frac{1}{k} y \in C$ for k sufficiently large. Let $D = \{\sum_{n=1}^{\infty} a_n e_n : \sum_{n=1}^{\infty} |a_n| \leq 1\}$. D is a closed convex set. [Use Fatou's Lemma to see that D is closed]. Hence $C \subseteq D$. We can write x as $\sum_{n=1}^{\infty} b_n e_n$ with $\sum_{n=1}^{\infty} |b_n| \leq 1$. Since $x + \frac{1}{k} y \in C$ we also have $\sum_{n=1}^{\infty} \left|b_n + \frac{1}{kn}\right| \leq 1$. It follows that $\sum_{n=1}^{\infty} \left|\frac{1}{kn}\right| \leq 1$ which is absurd.

Problem 449

Let $\{x_n\}$ converge to x weakly in l^2. Show that there is a subsequence $\{x_{n_j}\}$ which converges in the Cesaro sense in the norm, i.e. $\{\frac{x_{n_1} + x_{n_2} + ... + x_{n_k}}{k}\}$ converges in the norm. [Due to Banach and Saks]

We may assume that the weak limit of $\{x_n\}$ is 0. Let $M = \sup\{\|x_n\| : n = 1, 2, ...\}$. We pick inductively $n_1, n_2, ...$ as follows: $n_1 = 1$ and n_{k+1} is chosen such that $\left|< x_{n_i}, x_{n_{k+1}} >\right| \leq \frac{1}{k}$ for $1 \leq i \leq k$. Then $\left\|x_{n_1} + x_{n_2} + ... + x_{n_{k+1}}\right\| \leq \left\|(x_{n_1} + x_{n_{k+1}}) + (x_{n_2} + x_{n_{k+1}}) + ... + (x_{n_k} + x_{n_{k+1}})\right\| + (k-1)\left\|x_{n_{k+1}}\right\|$ and $\|x_{n_i} + x_{n1}\|^2 \leq$

$2M^2+\frac{2}{k}$ for each $i \leq k$. If $\|x_{n_1}+x_{n_2}+...+x_{n_k}\|^2 \leq kM^2+2k$ then $\|x_{n_1}+x_{n_2}+...+x_{n_k+1}\|^2 \leq kM^2+2k+M^2+2\operatorname{Re} < x_{n_1}+x_{n_2}+...+x_{n_k}, x_{n_{k+1}} > \leq (k+1)M^2+2k+2k\frac{1}{k} = (k+1)M^2+2(k+1)$ Hence, $\|x_{n_1}+x_{n_2}+...+x_{n_k}\|^2 \leq kM^2+2k$ for all k and $\left\|\frac{x_{n_1}+x_{n_2}+...+x_{n_k}}{k}\right\|^2 \leq \frac{kM^2+2k}{k^2} \to 0.$

Problem 450

Any bounded linear map between normed linear spaces is weak-weak continuous. Is the converse true?

First part follows from definition of weak topology. Converse is also true: let $T : X \to Y$ be weak-weak continuous. Let $y^* \in Y^*$. Then $T^{-1}\{y : |y^*(y)| < 1\}$ is a weak neighbourhood of 0 in X. Hence it is also a neighbourhood of 0 in the norm topology and so it contains $\{x : \|x\| < \delta\}$ for some $\delta > 0$. Thus $\|x\| < \delta$ implies $|y^*(Tx)| < 1$. This implies $|y^*(Tx)| < 2/\delta$ whenever $\|x\| \leq 1$. Hence $T\{x : \|x\| \leq 1\}$ is weakly bounded. This implies that $T\{x : \|x\| \leq 1\}$. [This is a simple application of Uniform Boundedness Principle: let $\|x_n\| \leq 1$ for each n and define $T_n : X^* \to K$ $(= \mathbb{R}$ or $\mathbb{C})$ by $T_n(x^*) = x^*(x_n)$. Then $\sup |T_n(x^*)| : n \geq 1\} < \infty$ for each x^* so $\sup\{\|T_n\| : n \geq 1\} < \infty$ which means $\{\|x_n\|\}$ is bounded].

Problem 451

Find two closed subspaces of a Hilbert space whose sum is not closed.

Let $T : l^2 \to l^2$ be any bounded operator whose range is dense but not equal to l^2. [Example $Tx = \{a_n x_n\}$ where $0 < a_n \to 0$ fast enough; $a_n = \frac{1}{n^2}$. for example]. Let H be the direct sum of l^2 with itself, $M = \{(x, Tx) : x \in l^2\}$ and $N = \{(x, 0) : x \in l^2\}$. Of course, M and N are closed subspaces of H. We claim that $H = [M+N]^-$ but $M+N \neq H$. If $b \notin T(l^2)$ then $(a, b) \notin M+N$. If (a, b) is orthogonal to $M+N$ then $< a, x > + < b, Tx >= 0$ and $< a, x > + < b, 0 >= 0$ for all x, y. Thus $a = 0$ and $< b, Tx >= 0$ for all x. Since the range of T is dense we get $b = 0$. Hence $H = [M+N]^-$.

Problem 452

Let X be an infinite dimensional Banach space with a Hamel basis $\{e_i\}$. Define $f_i : X \to K$ by $f_i(\sum a_j e_j) = a_i$. Show that f_i cannot be continuous for every i.

Let $x = \displaystyle\sum_n a_n e_{i_n}$ where $e_{i_n}'s$ are distinct and $a_n > 0$ are such that the series converges, e.g. $a_n = \frac{1}{2^n\|e_{i_n}\|}$. We can write x as a finite sum $\displaystyle\sum_{j \in F} b_j e_j$. where F is a finite set. Suppose $i_p \notin F$. Then f_{i_p} is not continuous. This is because $f_{i_p}(\sum_{n=1}^{r} a_n e_{i_n}) = a_p$ if $r \geq p, \sum_{n=1}^{r} a_n e_{i_n} \to x$ as $r \to \infty$ but $f_{i_p}(x) = f_{i_p}(\sum_{j \in F} b_j e_j) = 0$.

Problem 453

Show that any separable metric space is isometric to a subset of $C[0,1]$. (Assume Banach - Mazur Theorem)

Let $\{x_n\}$ be dense in the metric space (X,d). Define $f : X \to l^\infty$ by $f(x) = (d(x,x_n) - d(y,x_n))$ where $y \in X$ is fixed. Since $d(x,x_n) - d(y,x_n) \leq d(x,y)$ and $d(y,x_n) - d(x,x_n) \leq d(x,y)$ we see that f is a well-defined map into l^∞. Now $\|f(x) - f(z)\| = \sup\{|d(x,x_n) - d(z,x_n)| : n \geq 1\}$. Clearly $\|f(x) - f(z)\| \leq d(x,z)$. If $x_{n_j} \to z$ then $\|f(x) - f(z)\| \geq \left|d(x,x_{n_j}) - d(z,x_{n_j})\right| \to d(x,z)$. Hence (X,d) is isometric to a separable subset of l^∞. The closed subspace generated by the range of f is a separable Banach space. By Banach - Mazur Theorem there is an isometric isomorphism from this Banach space into $C[0,1]$. The composition of these two isometries gives an isometry from X into $C[0,1]$.

Problem 454

Show that sum of two closed subspaces of a Banach space need not be closed. Show that if M and N are closed subspaces of a Banach space X with $M \cap N = \{0\}$ then $M + N$ is closed if and only if $\inf\{\|x - y\| : x \in M, y \in N, \|x\| = 1 = \|y\|\} > 0$.

For the first part we actually give an example in a Hilbert space: take $M = \bar{sp}\{e_1, e_3, ...\}$ and $N = \bar{sp}\{e_1 + \frac{1}{2}e_2, e_3 + \frac{1}{2^2}e_4, ..., e_{2n-1} + \frac{1}{2^n}e_{2n}, ...\}$ in $X = l^2$. If $u_n = \sum_{i=1}^{n} \frac{1}{2^i}e_{2i}$. then $u_n \to u$ where $u = \sum_{i=1}^{\infty} \frac{1}{2^i}e_{2i}$. We claim that $u_n \in M + N$ for each n but $u \notin M + N$. Since $u_n = \sum_{i=1}^{n}(\frac{1}{2^i}e_{2i} + e_{2i-1}) + \sum_{i=1}^{n}(-1)e_{2i-1}$ we get $u_n \in M + N$. If $u \in M + N$ then we can write $\sum_{i=1}^{\infty} \frac{1}{2^i}e_{2i} = \sum_{i=1}^{\infty} a_i e_{2i-1} + \lim_{k\to\infty} \sum_{i=1}^{k} b_i^{(k)}(\frac{1}{2^i}e_{2i} + e_{2i-1})$. Taking $2j - th$ coordinates on both sides we get $\frac{1}{2^i} = \lim_{k\to\infty} b_i^{(k)} \frac{1}{2^i}$ or $\lim_{k\to\infty} b_i^{(k)} = 1$. Taking $(2j-1) - st$ coordinates we get $0 = a_i + \lim_{k\to\infty} b_i^{(k)} = a_i + 1$. Thus $a_i = -1$ for all i which contradicts the fact that $\sum_{i=1}^{\infty} a_i e_{2i-1}$ converges.

We now prove the second part. Suppose $M+N$ is closed. Define $P : M+N \to N$ by $P(m+n) = n$ for $m \in M, n \in N$. This is a well-defined linear map and it is continuous by Closed Graph Theorem. Let $x \in M, y \in N$ and $\|x\| = 1 = \|y\|$. Then $1 = \|y\| = \|P(x-y)\| < \|P\| \|x-y\|$ so $\|x-y\| \geq \frac{1}{\|P\|}$. For the converse part suppose $M+N$ is not closed. Then, for each n, we can find $x_n \in M, y_n \in N$ such that $n\|x_n - y_n\| < \|y_n\|$. [Otherwise there exists n such that $n\|x-y\| \geq \|y\|$ for all $x \in M$ and $y \in N$ which implies that if $x_j + y_j \to z$ with $\{x_j\} \subseteq M, \{y_j\} \subseteq N$ then $\|y_j - y_l\| \leq n\|(x_j+y_j)-(x_l+y_l)\| \to 0$ as $j, l \to \infty$ so $\{y_j\}$ converges to some $y \in N$. Clearly $x_j \to z - y = x$ (say) so $z = x+y \in M+N$. This means $M+N$ is closed, contrary to our assumption]. We have $\left\|\frac{x_n}{\|y_n\|} - \frac{y_n}{\|y_n\|}\right\| < 1/n$. This implies that $\left|1 - \left\|\frac{x_n}{\|y_n\|}\right\|\right| = \left|\left\|\frac{y_n}{\|y_n\|}\right\| - \left\|\frac{x_n}{\|y_n\|}\right\|\right| \leq \left\|\frac{y_n}{\|y_n\|} - \frac{x_n}{\|y_n\|}\right\| < 1/n$. In other words, $\left\|\frac{x_n}{\|y_n\|}\right\| \in (1-1/n, 1+1/n)$. Let $t_n = \left\|\frac{x_n}{\|y_n\|}\right\| = \frac{\|x_n\|}{\|y_n\|}$ and $u_n = \frac{x_n}{\|x_n\|}$. We have $\left\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\right\| < 1/n + \left\|\frac{x_n}{\|x_n\|} - \frac{x_n}{\|y_n\|}\right\|$ $= 1/n + \|x_n\| \left|\frac{1}{\|x_n\|} - \frac{1}{\|y_n\|}\right| = 1/n + \left|1 - \frac{\|x_n\|}{\|y_n\|}\right| = 1/n + |1-t_n| < 2/n$. This proves that $\inf\{\|x-y\| : x \in M, y \in N, \|x\| = 1 = \|y\|\} > 0$.

Problem 455

Let C be a closed convex set in a Hilbert space H. For any $x \in H$ let Px be the unique element of C such that $\|x - Px\| \leq \|x-y\|$ for all $y \in C$. Prove that $\|Px - Py\| \leq \|x-y\|$ $\forall x, y \in H$. Also show that $\|x\|^2 - \|x-Px\|^2$ is a convex function on H.

We have $\|x - Px\| \leq \|x - (ty + (1-t)Px)\|$ if $y \in C$ and $t \in [0,1]$. Hence $\|x-Px\|^2 \leq \|x-Px\|^2 + t^2\|y-Px\|^2 + 2t \operatorname{Re} < x - Px, Px - y >$. This gives $0 \leq t\|y-Px\|^2 + 2\operatorname{Re} < x-Px, Px-y >$. Hence $\operatorname{Re} < x-Px, Px-y > \geq 0$ for $y \in C$. Hence $\operatorname{Re} < x-Px, Px-Py > \geq 0$ for all x, y . Interchanging x and y in we get $\operatorname{Re} < y-Py, Py-Px > \geq 0$ so $\operatorname{Re} < Py-y, Px-Py > \geq 0$. Adding we get $\operatorname{Re} < x-Px+Py-y, Px-Py > \geq 0$ which gives us $\operatorname{Re} < x-y, Px-Py > - \|Px-Py\|^2 \geq 0$. Hence $\|Px-Py\|^2 \leq \|x-y\| \|Px-Py\|$ or $\|Px-Py\| \leq \|x-y\|$. Now let $\phi(x) = \|x\|^2 - \|x-Px\|^2$. Then $\phi(x) = \sup\{\|x\|^2 - \|x-y\|^2 : y \in C\} = \sup\{2\operatorname{Re} < x, y > - \|y\|^2 : y \in C\}$ and convexity is obvious from this.

Problem 456

Let $\{x_n\}, \{y_n\}$ be sequences in a normed linear space such that $x_n = y_n = 1$

for all n and $\|x_n + y_n\| \to 2$. Show that $\|t_n x_n + (1-t_n) y_n\| \to 1$ for any $\{t_n\} \subseteq [0,1]$.

Without loss of generality we assume that $\{t_n\}$ converges to some $t \in [0,1]$. Since $|\|t_n x_n + (1-t_n)y_n\| - \|t x_n + (1-t) y_n\|| \leq 2|t_n - t|$ it suffices to consider the case when $t_n = t$ for all n. Let $\phi_n(t) = \|t x_n + (1-t) y_n\|$. Then ϕ_n is convex. Also $\phi_n(0) = 1 = \phi_n(1)$ and $\phi_n(\frac{1}{2}) \to 1$. From these we can show that $\phi_n(t) \to 1$ for $0 \leq t \leq 1$: if $0 \leq t \leq \frac{1}{2}$ then $\phi_n(\frac{1}{2}) \leq \alpha\phi_n(1)+(1-\alpha)\phi_n(t)$ where α is defined by $\frac{1}{2} = \alpha + (1-\alpha)t$. Thus $\liminf \phi_n(t) \geq 1$ and since $\phi_n(t) \leq 1$ for all n we get $\lim \phi_n(t) = 1$. For $\frac{1}{2} \leq t \leq 1$ the result follows by interchanging x_n and y_n and replacing t by $1-t$.

Another proof: there exists $x_n^* \in X^*$ such that $\|x_n^*\| = 1$ and $x_n^*(\frac{x_n+y_n}{2}) = \left\|\frac{x_n+y_n}{2}\right\|$. Since $|x_n^*(x_n)| \leq 1$ and $|x_n^*(y_n)| \leq 1$ it follows that $x_n^*(x_n) \to 1$ and $x_n^*(y_n) \to 1$. Hence $x_n^*(t_nx_n + (1-t_n)y_n) \geq t_n(1-\varepsilon) + (1-t_n)(1-\varepsilon)$ for n sufficiently large. Since $x_n^*(t_nx_n + (1-t_n)y_n) \leq \|t_nx_n + (1-t_n)y_n\|$ we get $\liminf \|t_nx_n + (1-t_n)y_n\| \geq 1-\varepsilon$.

Problem 457

Let u be a unit vector in $\mathbb{R}^n$ and A be a Lebesgue measurable subset of $\mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$ we have $m(L_x) = 0$ where $L_x = \{t \in \mathbb{R} : x + tu \in A\}$. Show that $m_n(A) = 0$ (m_n is Lebesgue measure on $\mathbb{R}^n$ and m is Lebesgue measure on $\mathbb{R}$).

Let T be an isometric isomorphism of $\mathbb{R}^n$ such that $Tu = e_1$. If $t_2, t_3, ..., t_n \in \mathbb{R}^{n-1}$ then $\{t \in \mathbb{R} : (t, t_2, t_3, ..., t_n) \in T(A)\} = \{t \in \mathbb{R} : tTu + (0, t_2, t_3, ..., t_n) \in T(A)\} = \{t \in \mathbb{R} : tu + T^{-1}(0, t_2, t_3, ..., t_n) \in A\}$. The hypothesis with $x = T^{-1}(0, t_2, t_3, ..., t_n)$ shows $m\{t \in \mathbb{R} : (t, t_2, t_3, ..., t_n) \in T(A)\} = 0$. Since $t_2, t_3, ..., t_n$ are arbitrary Fubini's Theorem shows that $m_n(T(A)) = 0$. But $m_n(T(A)) = \det(T)m_n(A)$ and $\det(T) \neq 0$ so $m_n(A) = 0$.

Problem 458

Let A be a subset of a Banach space X and $\beta \in (0,1)$ Suppose we have the following property: for any $x \in A$ and any $\delta > 0$ there exists $y \in X$ such that $\|y - x\| \leq \delta$ and $B(y, \beta \|y - x\|) \cap A = \emptyset$. Show that A is a nowhere dense set and it has Lebesgue measure of measure 0 when $X = \mathbb{R}^n$.

Remark: a set A with the property stated above is called 'porous'.

If possible let $\bar{A}$ have an interior point x. Let $\{x_n\} \subseteq A$ and $x_n \to x$. By hypothesis there exists y_n such that $\|y_n - x_n\| \leq 1/n$ and $B(y_n, \beta \|y_n - x_n\|) \cap A = \emptyset$. Let $B(x, \rho) \subseteq \bar{A}$. If $\|z - y_n\| \leq \beta \|y_n - x_n\|$ then $\|z - x\| \leq \beta \|y_n - x_n\| + \|y_n - x_n\| + \|x_n - x\|$. Hence if we choose n so large that $\beta \|y_n - x_n\| + \|y_n - x_n\| + \|x_n - x\| < \rho$ we get $\|z - x\| < \rho$ and $z \in B(x, \rho) \subseteq \bar{A}$. It follows that $B(y_n, \beta \|y_n - x_n\|) \subseteq \bar{A}$. This is clearly a contradiction to $B(y_n, \beta \|y_n - x_n\|) \cap A = \emptyset$. Hence A is nowhere dense. Now suppose $X = \mathbb{R}^n$ and $m_n(A) > 0$. There exists $x \in A$ such that $\frac{m(B(x,r)\cap A)}{m(B(x,r))} \to 1$ as $r \to 0$. Choose $r > 0$ so small that $m_n(B(x,t) \cap A) > (1 - (\frac{\beta}{2})^n)m(B(x,t))$ if $0 < t < r$. Choose $y \in X$ such that $\|y - x\| < r/2$ and $B(y, \beta \|y - x\|) \cap A = \emptyset$. We now have $(1 - (\frac{\beta}{2})^n)m_n(B(x, 2\|y - x\|)) < m_n(A \cap B(x, 2\|y - x\|))$. Since $B(y, \beta \|y - x\|) \cap$

$A = \emptyset$ we get $(1-(\frac{\beta}{2})^n) m_n(B(x, 2\|y-x\|)) < m_n(B(x, 2\|y-x\|) \backslash B(y, \beta\|y-x\|))$. Noting that $B(y, \beta\|y-x\|) \subseteq B(x, 2\|y-x\|)$ we can compute $m_n(B(x, 2\|y-x\|) \backslash B(y, \beta\|y-x\|))$ in terms of the measure α of the open ball of radius 1 around 0: $m_n(B(x, 2\|y-x\|) \backslash B(y, \beta\|y-x\|)) = 2^n \|y-x\|^n \alpha - \beta^n \|y-x\|^n \alpha$. Finally we have $(1-(\frac{\beta}{2})^n) m_n(B(x, 2\|y-x\|)) \equiv (1-(\frac{\beta}{2})^n) 2^n \|y-x\|^n \alpha < 2^n \|y-x\|^n \alpha - \beta^n \|y-x\|^n \alpha$. This is a contradiction.

Problem 459

Do there exists two dense subspaces of a Hilbert space whose intersection is $\{0\}$?

Yes. Step functions and C^∞ functions with compact support in $L^2(\mathbb{R})$.

Problem 460

Let X be a Banach space and K be a bounded closed convex set in X. If every continuous map from K into itself has a fixed point show that K is compact.

Suppose not. Then there exists $\delta > 0$ such there is no finite δ - net for K. [i.e. a finite number of balls of radius cannot cover K]. Let $x_0 \in K$. There exists $x_1 \in K$ such that $d(x_1, span\{x_0\}) > \delta/2$. [Suppose $d(x, span\{x_0\}) \leq \delta/2$ for all $x \in K$. Since K is bounded we get a bounded set $S \subseteq span\{x_0\}$ such that $d(x, span\{x_0\}) \leq \delta/2$. S can be covered by a finite number of balls of radius $\delta/2$ and hence K itself has a δ -net]. Having chosen x_i for $0 \leq i \leq k$ choose x_{k+1} in K such that $d(x_{k+1}, span\{x_0, x_1, ..., x_k\}) > \delta/2$. [Existence x_{k+1} can be proved as above since bounded subsets of $span\{x_0, x_1, ..., x_k\}$ are totally bounded]. By induction we get a sequence $\{x_n\} \subseteq K$ such that $d(x_{k+1}, span\{x_0, x_1, ..., x_k\}) > \delta/2$ for all k. In view of the convexity of K each of the segments $[x_i, x_{i+1}]$ is contained in K. There exists a homeomorphism $\phi : L \equiv \bigcup_{i=0}^{\infty} [x_i, x_{i+1}] \to [0, \infty)$. [The segments $[x_i, x_{i+1}]$ (which are disjoint except for the fact that adjacent segments have one point in common) can be mapped homeomorphically to $[i, i+1]$ via $tx_i + (1-t)x_{i+1} \to ti + (1-t)(i+1)$ and these can be combined to get ϕ]. Note that $L \subseteq K$. We claim that L is closed. Suppose $\{y_n\} \subseteq L$ and $y_n \to y$. By going to a subsequence we may suppose $y_n = t_n x_{i_n} + (1-t_n) x_{i_n+1}$ with $\{t_n\}$ converging to some t. In view of boundedness of K it follows that $z_n \equiv t x_{i_n} + (1-t) x_{i_n+1}$ also converges to y. Since finite unions of $[x_i, x_{i+1}]$ are closed we may suppose $i_n \uparrow \infty$. Choose n so large that $\|z_n - z_{n+1}\| < (\delta/2)(1-t)$. Then $\left\| \frac{t}{1-t} x_{i_n} + x_{i_n+1} - \{ \frac{t}{1-t} x_{i_{n+1}} + x_{i_{n+1}+1} \} \right\| < \delta/2$. But then the distance from $x_{i_{n+1}+1}$ and $span\{x_j : j < i_{n+1} + 1\}$ is less than $\delta/2$ which is a contradiction.

Now define $g : L \to L$ by $g(x) = \phi^{-1}(\phi(x) + 1)$ and define $f : K \to K$ by $f(x) = g \circ \theta$ where θ is a continuous map from K into L which is identity on L. The existence of θ can be proved using Tietze Extension Theorem. [A continuous map from a closed subset of a metric space into $[0,\infty)$ can be extended to a continuous map from the metric space into the same interval: just take the extension ξ into $\mathbb{R}$ given by Tietze Theorem and consider $|\xi|$]. If $f(x) = x$ then x belongs to the range of g which if L and hence $\theta(x) = x$. Thus $g(x) = x$. But this is absurd since $\phi(g(x)) = \phi(x) + 1 \neq \phi(x)$. This completes the proof.

Problem 461

Let X be a Banach space and K be a closed convex set in X. If every continuous map from K into itself has a fixed point show that K is compact.

[Boundedness of K has been dropped from previous problem]

Assume, w.l.o.g. that $0 \in K$. Assume that K is not compact. Suppose there exists a closed bounded convex set $K_1 \subseteq K$ which is not compact. In this case K_1 contains a homeomorphic copy of $[0,\infty)$ and we can proceed as in above problem to complete the proof. In the contrary case $\{x \in K : \|x\| \leq 1\}$ is necessarily compact. Since K is unbounded, convex and contains 0 we can find $\{x_n\} \subseteq K$ such that $\|x_n\| = n$ for all n. Compactness of $\{x \in K : \|x\| \leq 1\}$ ensures that there is a subsequence $\{n_j\}$ of $\{1,2,...\}$ and y such that $\frac{x_{n_j}}{n_j} \to y$. Claim: $[0,\infty)y \subseteq K$. Let $t \in [0,\infty)$. Since x_{n_j} and $0 \in K$ and K is convex, $t\frac{x_{n_j}}{n_j} \in K$ for all j sufficiently large. Hence $ty \in K$ and the claim is proved. Now we can repeat the proof of previous problem again.

Problem 462

Show that Let X be a Banach space and K be a closed convex set in X. If every continuous map from K into itself has a fixed point show that K is compact.

[Boundedness of K has been dropped from previous problem]

Assume, w.l.o.g. that $0 \in K$. Assume that K is not compact. Suppose there exists a closed bounded convex set $K_1 \subseteq K$ which is not compact. In this case K_1 contains a homeomorphic copy of $[0,\infty)$ and we can proceed as in above problem to complete the proof. In the contrary case $\{x \in K : \|x\| \leq 1\}$ is necessarily compact. Since K is unbounded, convex and contains 0 we can find $\{x_n\} \subseteq K$ such that $\|x_n\| = n$ for all n. Compactness of $\{x \in K : \|x\| \leq 1\}$ ensures that there is a subsequence $\{n_j\}$ of $\{1,2,...\}$ and y such that $\frac{x_{n_j}}{n_j} \to y$. Claim: $[0,\infty)y \subseteq K$. Let $t \in [0,\infty)$. Since x_{n_j} and $0 \in K$ and K is convex, $t\frac{x_{n_j}}{n_j} \in K$ for all j sufficiently large. Hence $ty \in K$ and the claim is proved. Now we can repeat the proof of previous problem again.

Problem 463

Show that $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is not homeomorphic to $\{x \in \mathbb{R}^n : \|x\| = 1\}$.

Remark: if X is an infinite dimensional Banach space that $X, \{x \in X : \|x\| \leq 1\}$ and $\{x \in X : \|x\| = 1\}$ are all homeomorphic(and they are homeomorphic to $\mathbb{R}^\infty$)! [cf. Bessaga and Pelczynski, Selected Topics in Infinite Dimensional Topology]

Suppose $\phi : \{x \in \mathbb{R}^n : \|x\| \leq 1\} \to \{x \in \mathbb{R}^n : \|x\| = 1\}$ is a homeomorphism. Consider $\phi^{-1}(\phi(-x))$. This is a continuous map of $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ into itself. By Brouwer's Fixed Point Theorem there exists x such that $\phi^{-1}(\phi(-x)) = x$. But then $\phi(x) = \phi(-x)$ contradicting the fact that ϕ is a homeomorphism.

Problem 464

Let C be a closed convex set in a Banach space X. If $f : C \to C$ is a continuous map such that $f(C)$ is compact show that f has a fixed point.

We use Schauder's Fixed Point Theorem and the fact that the closed convex hull of a compact set is compact. [My notes on Fixed Point Theorems and Theorem 3.25 of Rudin;s Functional Analysis]. Let K be the closed convex hull of $f(C)$. Then K is a compact convex set and $K \subseteq C$. Also $f(K) \subseteq f(C) \subseteq K$. By schauder's theorem f has a fixed point in K, hence in C.

Problem 465

Let C be a closed convex bounded set in a Banach space X. Let $f : C \to C$ be a map such that $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in C$. Prove that $\inf\{\|f(x) - x\| : x \in C\} = 0$.

Remark: if C is also compact then we can conclude that f has a fixed point.

Let $f_n(x) = \frac{1}{n}x_0 + (1 - \frac{1}{n})f(x)$ where $x_0 \in C$ is fixed. Then f_n maps C into itself and $\|f_n(x) - f_n(y)\| \leq (1 - \frac{1}{n}) \|x - y\|$. This implies that $f_n(x_n) = x_n$ for some $x_n \in C$. [Fix n and denote by $f_n^{(k)}$ the k- fold iteration of f_n with itself. For $j < k$, $\left\|f_n^{(k)}(x_0) - f_n^{(j)}(x_0)\right\| \leq (1 - \frac{1}{n})^j \left\|f_n^{(k-j)}(x_0) - x_0\right\|$ and $\left\|f_n^{(k-j)}(x_0) - x_0\right\| \leq 2\sup\{\|z\| : z \in C\}$ so $\{f_n^{(k)}(x_0)\}_k$ is Cauchy. Its limit x_n satisfies $f_n(x_n) = x_n$]. Thus$\frac{1}{n}x_0 + (1 - \frac{1}{n})f(x_n) = x_n$ for all n. Now $\|f(x_n) - x_n\| = \left\|f(x_n) - \frac{1}{n}x_0 - (1 - \frac{1}{n})f(x_n)\right\| \leq \frac{\|x_0\|}{n} + \frac{\|f(x_n)\|}{n}$. Since $\{f(x_n\} \subseteq C$ and C is bounded it follows that $\|f(x_n) - x_n\| \to 0$.

Problem 466

In previous problem show that f need not have a fixed point.

Let $C = \{f \in C[0,1] : f(0) = 0, f(1) = 1 \text{ and } f([0,1]) \subseteq [0,1]\}$. Let $F : C \to C$ be defined by $F(f)(x) = xf(x)$. Then $\|F(f) - F(g)\| < \|f - g\|$ if $f \neq g$. [If $\|F(f) - F(g)\| = \|f - g\|$ then there exists x_0 such that $\|f - g\| =$

$\|F(f)-F(g)\| = |x_0 f(x_0) - x_0 g(x_0)| \le |f(x_0)-g(x_0)| \le \|f-g\|$ so we must have equality throughout. Hence $x_0 = 1$ and $\|f-g\| = |x_0 f(x_0) - x_0 g(x_0)| = |1-1| = 0]$. If $f \in C$ and $F(f) = f$ then $xf(x) = f(x)$ for all x. But then $f(x) = 0$ for $x < 1$ making f discontinuous at 1.

Problem 467

Let K be a compact convex set in a Banach space having at least two points. Show that K has a non-diametral point, i.e., there exists $x \in K$ such that $\sup\{\|x-y\| : y \in K\} < d$ where d is the diameter of K.

Suppose $\sup\{\|x-y\| : y \in K\} < d$ for all $x \in K$. Let $x_1 \in K$. There exists $x_2 \in K$ such that $\|x_1 - x_2\| = d$. Having found $x_1, x_2, ..., x_n$ we can choose $x_{n+1} \in K$ such that $\left\|x_{n+1} - \frac{x_1+x_2+...+x_n}{n}\right\| = d$. We get a contradiction by showing that the sequence $\{x_n\}$ in K has no convergent subsequence. We have $d = \left\|x_{n+1} - \frac{x_1+x_2+...+x_n}{n}\right\| \le \frac{\|x_{n+1}-x_1\|+\|x_{n+1}-x_2\|+...+\|x_{n+1}-x_n\|}{n} \le \frac{d+d+...+d}{n} = d$ which implies $\|x_{n+1} - x_i\| = d$ for $i \le n$. This is true for each n so $\|x_i - x_j\| = d$ whenever $i \ne j$.

Problem 468

Let X be a separable Banach space. Show that there exists a compact set K such that $\|x\| = 1$ for all $x \in K$ and X is the closed subspace spanned by K.

Let $\{x_n\}$ be dense in X. We may suppose $x_n \ne 0$ for each n. Let $M_n = span\{x_1, x_2, ..., x_n\}$. Consider the sequence $\{\frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|} + \frac{x_2}{2\|x_2\|}, \frac{x_1}{\|x_1\|} + \frac{x_2}{2\|x_2\|} + \frac{x_2}{2^2\|x_2\|}, ...\}$. Call this sequence $\{y_n\}$. It is clear that $y_n \to y = \sum_{n=1}^{\infty} \frac{x_n}{2^n\|x_n\|}$. Let $z_n = \frac{y_n}{\|y_n\|}$ and $z = \frac{y}{\|y\|}$. Then $z_n \to z$. Let $K = \{z\} \cup \{z_1, z_2, ...\}$. Then K is a compact set each of whose elements has norm 1. Clearly, $M_n \subseteq span\{y_1, y_2, .., y_n\} = span\{z_1, z_2, .., z_n\}$. Now $X \subseteq [\bigcup_n M_n]^- \subseteq [span(K)]^-$ and hence X is the closed subspace spanned by K.

Problem 469

Prove or disprove the following: if μ_n, μ are probability measures on a compact Hausdorff space Ω such that $\mu_n \xrightarrow{w} \mu$ then there exists a finite positive measure ν on Ω such that $\sup\{\mu_n(E) : n \ge 1\} \to 0$ as $\nu(E) \to 0$.

Remark: if we assume that $\{\mu_n\}$ converges weakly in $C^*(\Omega)$, i.e. $\Phi(\mu_n) \to \Phi(\mu)$ $\forall \Phi \in C^{**}(\Omega)$ then there does exist ν with above property. [Theorem 13.43 of Banach Space Theory by Fabian et al].

False: let $\Omega = [0,1], \mu_n = \delta_{1/n}, \mu = \delta_0$. Since $\sum \nu\{\frac{1}{j}\} < \infty$ we get $\nu\{\frac{1}{j}\} \to 0$ as $j \to \infty$. However $\sup\{\mu_n\{\frac{1}{j}\} : n \ge 1\} = 1$ for each j.

Problem 470

Prove or disprove: if Ω is a compact Hausdorff space, $\{f\}\cup\{f_n\} \subseteq C(\Omega)$ and $\{P_n\}$ is a sequence of Borel probability measures on Ω such that $\int f_n d\mu \to \int f d\mu$ for every Borel probability measure μ on Ω and $\int f dP_n \to \int f dP$ for every $f \in C(\Omega)$ then $\int f_n dP_n \to \int f dP$.

Remark: if we assume that $\{P_n\}$ converges weakly in $C^*(\Omega)$, i.e. $\Phi(P_n) \to \Phi(P)$ $\forall \Phi \in C^{**}(\Omega)$ then it is true that $\int f_n dP_n \to \int f dP$. [Theorem 13.43 of Banach Space Theory by Fabian et al].

False: let $\Omega = [0,1], P_{1/n} = \delta_n, P = \delta_0, f(x) \equiv 0$ and $f_n(x) = \begin{cases} nx \text{ for } 0 \leq x \leq 1/n \\ n(\frac{2}{n} - x) \text{ for } 1/n \leq x \leq 2/n \\ 0 \text{ for } 2/n \leq x \leq 1 \end{cases}$.

Problem 471

Let Ω be a compact Hausdorff space and $x \in \Omega$. Show that there is a countable base of neighbourhoods of x if and only if $\{x\}$ is a G_δ set.

If $\{U_n\}$ is a countable base of neighbourhoods of x then $\{x\} = \bigcap\limits_n U_n$ is a G_δ. Conversely, suppose $\{x\} = \bigcap\limits_n U_n$ with each U_n open. There exists open sets V_n such that $\times \in V_n \subseteq \bar{V_n} \subseteq U_n$. Let $W_n = V_1 \cap V_2 \cap ... \cap V_n$. Each W_n is a neighbourhood of x. Let U be any neighbourhood of x. If U does not contain any W_n then there exist points x_n in $W_n \backslash U, n = 1, 2, ...$ The sets $\bar{W_n} \backslash U, n = 1, 2, ...$ are decreasing, compact and non-empty. By compactness of Ω, $\bigcap\limits_n \{\bar{W_n} \backslash U\}$ is non-empty. However $\bigcap\limits_n \{\bar{W_n} \backslash U\} \subseteq \bigcap\limits_n \{\bar{V_n} \backslash U\} \subseteq \bigcap\limits_n \{U_n \backslash U\} = \{x\} \backslash U = \emptyset$. Hence every neighborhood U of x contains one of the sets $\{W_n\}$.

Problem 472

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear map. Show that T is compact if and only if there is a sequence $\{x_n^*\} \subseteq X^*$ such that $\|Tx\| \leq \sup\{|x_n^*(x)| : n \geq 1\}$ and $\|x_n^*\| \to 0$.

Suppose $\{x_n^*\} \subseteq X^*, \|Tx\| \leq \sup\{|x_n^*(x)| : n \geq 1\}$ and $\|x_n^*\| \to 0$. Let $\Omega = \{0\} \cup \{x_n^* : n \geq 1\}$. With the metric from X^* this set is a compact metric space. Let $\{x_j\}$ be a sequence in the closed unit ball of X. Define a sequence $\{f_j\}$ in $C(\Omega)$ by $f_j(x_n^*) = x_n^*(x_j)$ and $f_j(0) = 0$. It is trivial to check that this sequence is uniformly bounded and equicontinuous on Ω. Hence there is a subsequence $\{f_{j_l}\}$ converging uniformly on Ω to some $f \in C(\Omega)$. Hence $\sup\{|f_{j_l}(x_n^*) - f_{j_r}(x_n^*)| : n \geq 1\} \to 0$ as $n, r \to \infty$. This means $\sup\{|x_n^*(x_{j_l}) - x_n^*(x_{j_r})| : n \geq 1\}$. But then $\|T(x_{j_l}) - T(x_{j_r})\| \leq \sup\{|x_n^*(x_{j_l} - x_{j_r})| :$

$n \geq 1\} \to 0$ proving that $\{T(x_n)\}$ has a convergent subsequence. For the converse let T be compact and $A = \{T^*(y^*) : \|y^*\| \leq 1\}$. Since T^* is compact too, A is relatively compact, hence totally bounded. Claim: there exists $\{x_n^*\} \subseteq X^*$ such that $\|x_n^*\| \to 0$ and $A \subseteq \bar{co}\{x_n^* : n \geq 1\}$. Once this claim is established we get $\|Tx\| = \sup\{|y^*(Tx)| : \|y^*\| \leq 1\} = \sup\{|(T^*y^*)(x)| : \|y^*\| \leq 1\}$

$= \sup\{|z^*(x)| : z^* \in A\} \leq \sup\{|x_n^*(x)| : n \geq 1\}$ thereby completing the proof. Let $A_1 = A$ and B_1 be a $\frac{1}{4}$ net for A (i.e. B_1 is a finite subset of A such that every point of A is at distance less than $\frac{1}{4}$ from some member of B_1). Let $A_2 = (A_1 - B_1) \cap \{x^* : \|x^*\| \leq \frac{1}{4}\}$. Having defined A_i, B_i for $1 \leq i \leq m$ let $A_{m+1} = (A_m - B_m) \cap \{x^* : \|x^*\| \leq \frac{1}{4^m}\}$ and B_{m+1} be a $\frac{1}{4^{m+1}}$ net for A_{m+1}. Let $\{x_n^*\}$ be obtained by first listing all the elements of $2B_1$, then all the elements of $2^2B_2, ...$ Since $B_n \subset A_n \subseteq \{x^* : \|x^*\| \leq \frac{1}{4^{n-1}}\}$ for $n \geq 2$ we get $\|x_n^*\| \to 0$. Now let $x^* \in A$. By definition of B_1 there exists $z_1^* \in B_1$ such that $\|x^* - z_1^*\| < \frac{1}{4}$. Hence $u_1^* \equiv x^* - z_1^* \in A_2$ and $x^* = u_1^* + z_1^*$. Now there exists $z_2^* \in B_2$ such that $\|z_2^* - z_1^*\| < \frac{1}{4^2}$. Now $u_2^* \equiv z_1^* - z_2^* \in A_2$ and $z_1^* = u_2^* + z_2^*$. Thus $x^* = u_1^* + z_1^* = u_1^* + u_2^* + z_2^*$. Proceeding like this we get $x^* = u_1^* + u_2^* + ... + u_j^* + z_j^*$ for each j. Since $z_j^* \in B_j \subseteq A_j$ we have $\left\|z_j^*\right\| \leq \frac{1}{4^{j-1}}$. Hence $x^* = \sum_{k=1}^{\infty}(2^k u_k^*)/2^k$. By the definition of $\{x_n^*\}$ it is clear that this (norm convergent) sum belongs to the closed convex hull of $\{x_n^* : n \geq 1\}$. This proves the claim.

Problem 473

Let $P, P_1, P_2, ...$ be Borel probability measures on $\mathbb{R}$ such that $P_n((a,b)) \to P((a,b))$ whenever $-\infty < a < b < \infty$. Show that $P_n \xrightarrow{w} P$.

Let $\varepsilon > 0$ and choose a positive number Δ such that $P((-\Delta, \Delta)) > 1 - \varepsilon$. There exists n_0 such that $P_n((-\Delta, \Delta)) > 1 - \varepsilon$ for $n \geq n_0$. Now $P_n((-\infty, a]) = P_n((-\infty, -\Delta]) + P_n((-\Delta, a]) < \varepsilon + P_n((-\Delta, a]) \leq \varepsilon + P_n((-\Delta, a + \delta)) \to \varepsilon + P((-\Delta, a + \delta))$

$\leq \varepsilon + P((-\infty, a + \delta))$ so $\limsup P_n((-\infty, a]) \leq \varepsilon + P((-\infty, a + \delta))$. Since $\delta > 0$ is arbitrary we get $\limsup P_n((-\infty, a]) \leq \varepsilon + P((-\infty, a])$. On the other hand $P_n((-\infty, a]) \geq P_n((-\Delta, a]) \geq P_n((-\Delta, a - \delta)) \to P((-\Delta, a - \delta)) > P((-\infty, a - \delta)) - \varepsilon$. Letting $\delta \to 0$ (and then $\varepsilon \to 0$) we get $\liminf P_n((-\infty, a]) \geq P((-\infty, a))$. It follows that $P_n((-\infty, a]) \to P((-\infty, a])$ whenever $p\{a\} = 0$.

Problem 474

In Problem 473 above can we conclude that $P_n(E) \to P(E)$ for every Borel set E?

No. Let $\{X_n\}$ be i.i.d. random variable taking values 0 and 1 with probability $\frac{1}{2}$ each and $S_n = \frac{X_1 + X_2 + ... + X_n - \frac{n}{2}}{(1/2)\sqrt{n}}$. By Central Limit Theorem $S_n \xrightarrow{d} Y$

where Y has standard normal distribution. Let $E = \mathbb{R}\backslash\{\frac{k-\frac{n}{2}}{(1/2)\sqrt{n}} : 0 \leq k \leq n, n \geq 1\}$. Denoting the distributions of S_n and Y by P_n and P respectively we see that the hypothesis of previous problem is satisfied (because any open interval (a,b) is a continuity interval for P) but $P(E) = 1 \neq 0 = \lim P_n(E)$.

Problem 475

Let T_n be a bounded operator on $C[0,1]$ for each n. Suppose $T_n f \geq 0$ whenever $f \geq 0, T_n 1 \to 1, T_n x \to x$ and $T_n x^2 \to x^2$ in the norm. Show that $T_n f \to f$ for all $f \in C[0,1]$.

[This result is due to Korovkin]

Fix $t \in [0,1]$ and $\varepsilon > 0$. Let p be a polynomial such that $\|p - f\|_\infty < \varepsilon/4$. Let m and M be the minimum and maximum of p'' on $[0,1]$. Let $\phi_1(x) = p(t) + (x-t)p'(t) + m\frac{(x-t)^2}{2}, \phi_2(x) = p(t) + (x-t)p'(t) + M\frac{(x-t)^2}{2}$. By Taylor's formula we have $\phi_1 \leq p \leq \phi_2$ on $[0,1]$. Note that $T_n\phi_i \to \phi_i, i = 1,2$ by hypothesis. Since $\phi_1(x) - \varepsilon/4 < f(x) < \phi_2(x) + \varepsilon/4$ for every x we have $T_n\phi_1 - (\varepsilon/4)T_n 1 \leq T_n f \leq T_n\phi_2 + (\varepsilon/4)T_n 1$. Note that $T_n\phi_1 - (\varepsilon/4)T_n 1 \to \phi_1 - (\varepsilon/4)$ and $T_n\phi_2 + (\varepsilon/4)T_n 1 \to \phi_2 + (\varepsilon/4)$. If $\delta > 0$ is sufficiently small, then $|\phi_2 - \phi_1| < \varepsilon/4$ on $I \equiv [t-\delta, t+\delta]$. Hence $\phi_1(x) - (\varepsilon/4) > p(x) - (\varepsilon/2) > f(x) - (3\varepsilon/4)$ and, similarly, $\phi_2 + (\varepsilon/4) < f(x) + \phi_2 + (3\varepsilon/4)$. It follows that $f(x) - (3\varepsilon/4) < T_n f(x) < f(x) + (3\varepsilon/4)$ for all $x \in I$ for n sufficiently large. It is now clear from compactness of $[0,1]$ that $T_n f \to f$ uniformly on $[0,1]$.

Problem 476

Let K be a subset of a separable Banach space X such that $\{x_n^*\} \subseteq X^*$ and $x_n^*(x) \to 0$ for all $x \in X$ imply $x_n^*(x) \to 0$ uniformly for $x \in K$. Show that K is relatively compact and conversely.

Suppose not. Then $\exists\ \delta > 0$ such that there is no δ - net for K. We can construct a sequence $\{x_n\}$ in K such that $d(x_{n+1}, span\{x_1, x_2, ..., x_n\}) > \delta/2$ for all n. Such a sequence was constructed in Problem 460 above. There exists x_n^* such that $\|x_n^*\| = 1, x_n^*(x_n) > \delta/2$ and $x_n^*(x_k) = 0$ for $1 \leq k \leq n-1$. [Define $x_n^*(\sum_{i=1}^n a_i x_i) = a_n\alpha_n$ where $\alpha_n = \frac{1}{\sup\{|a_n| : \left\|\sum_{i=1}^n a_i x_i\right\| \leq 1\}}$. x_n^* is well defined on $span\{x_1, x_2, ..., x_n\}$; it is easy to see that $\|x_n^*\| = 1$ and $|a_n| < \delta$, hence $x_n^*(x_n) > \delta/2$. Extend x_n^* using Hahn-Banach Theorem]. Since the closed unit ball of X^* is weak* compact metric (by separability of X) there is a subsequence $\{x_{n_j}^*\}$ of $\{x_n^*\}$ converging to some x^*. Since $x_n^*(x_k) = 0$ for $k < n$ we get $x^*(x_k) = 0$ for all k. Thus $(x_{n_j}^* - x^*)(x) \to 0$ as $j \to \infty$ for each x but the convergence is not uniform on K because $(x_{n_j}^* - x^*)(x_{n_j}) = x_{n_j}^*(x_{n_j}) > \delta/2$ for all j. The converse is easy: by Uniform Boundedness Principle $\{\|x_n^*\|\}$ is bounded. Since K is totally bounded it is easily seen that $x_n^*(x) \to 0$ uniformly for $x \in K$.

Problem 477

Let μ be a finite positive (non-zero) measure on a compact abelian topological group G such that $\mu * \mu = \mu$. Show that μ is a Haar measure. What happens if G is replaced by $\mathbb{R}$?

First note that $(\mu(G))^2 = \mu(G)$ so μ is a probability measure. Let f be a non-negative continuous function and $g(x) = \int f(yx)d\mu(y)$. Then $g(x) = \int f(yx)d(\mu*\mu)(y) = \int\int f(yzx)d\mu(z)d\mu(y) = \int g(yx)d\mu(y)$. Let $S = \{x \in G : g(x) = \sup\{g(z) : z \in G\}$. Let $x \in S$. Then $g(x) = \int g(yx)d\mu(y) \leq \sup\{g(z) : z \in G\} = g(x)$ so $\mu(S) = 1$. Since μ has full support and g is continuous it follows that g is a constant. Thus $\int f(yx)d\mu(y) = \int f(ye)d\mu(y)$ so $\int f d\mu_x = \int f d\mu$. This holds for all non-negative continuous functions f hence for all continuous function f and it follows that $\mu_x = \mu$. This holds for all x and we are done. If G is replaced by $\mathbb{R}$ then necessarily $\mu = \delta_0$: $\int e^{itx}d\mu(x) = (\int e^{itx}d\mu(x))^2$ so $\int e^{itx}d\mu(x) = 0$ or 1 for each t. By continuity we get $\int e^{itx}d\mu(x) = 1 = \int e^{itx}d\delta_0(x)$ for all t proving that $\mu = \delta_0$.

Problem 478

Prove or disprove:

a) there exist non-zero sequences $\{a_n\}_{n\in\mathbb{Z}}, \{b_n\}_{n\in\mathbb{Z}}$ such that $\sum\limits_{n=-\infty}^{\infty} |a_n| < \infty$, $\sum\limits_{n=-\infty}^{\infty} |b_n| < \infty$ and $\sum\limits_{n=-\infty}^{\infty} a_{m-n}b_n = 0$ for all $m \in \mathbb{Z}$

b) there exists a non-zero sequence $\{a_n\}_{n\in\mathbb{Z}}$ such that $\sum\limits_{n=-\infty}^{\infty} |a_n| < \infty$ and $\sum\limits_{n=-\infty}^{\infty} a_{m-n}a_n = 0$ for all $m \in \mathbb{Z}$

a) True. Let f and g be smooth functions $: \mathbb{R} \to \mathbb{R}$ with disjoint supports contained in $(0, 2\pi)$. Since f' and g' are of bounded variation we have $\left|(f')\hat{}\,(n)\right| \leq \frac{C}{|n|}$ and $\left|(g')\hat{}\,(n)\right| \leq \frac{C}{|n|}$ for some $C < \infty$ for $n \neq 0$. This gives $\left|\hat{f}(n)\right| \leq \frac{C}{n^2}$ and $|\hat{g}(n)| \leq \frac{C}{n^2}$ for $n \neq 0$. It follows that $\sum\limits_{n=-\infty}^{\infty} |a_n| < \infty$ and $\sum\limits_{n=-\infty}^{\infty} |b_n| < \infty$ where $a_n = \hat{f}(n)$ and $b_n = \hat{g}(n)$. Let $c_n = \sum\limits_{n=-\infty}^{\infty} a_{m-n}b_n$. Then $\sum\limits_{n=-\infty}^{\infty} |c_n| < \infty$ and a Fubini argument shows that $\sum\limits_{n=-\infty}^{\infty} c_n e^{inx} = (\sum\limits_{n=-\infty}^{\infty} a_n e^{inx})(\sum\limits_{n=-\infty}^{\infty} b_n e^{inx}) = f(x)g(x) = 0$ for all x. [We have used the fact that the Fourier series of a differentiable function converges to the function at each point]. From uniqueness of Fourier coefficients it follows that $c_n = 0$ for all n.

b) False. With $f(x) = \sum\limits_{n=-\infty}^{\infty} a_n e^{inx}$ we would have $f^2 \equiv 0$ which implies $a_n = 0$ for all n.

Next few problems are selected from Davidson and Donsig's book on Real Analysis and Applications.

Problem 479

Let M and N be closed subspaces of a real Banach space X such that $M \cap N = \{0\}$. Prove that $M + N$ is closed if and only if $\inf\{\|x - y\| : x \in M, y \in N, \|x\| = 1 = \|y\|\} > 0$.

Suppose $M + N$ is closed and $P : M + N \to M$ be the projection map. Closed Graph Theorem shows that P is continuous. If $\|x_n - y_n\| \to 0$ with $x_n \in M, y_n \in N, \|x_n\| = 1 = \|y_n\|$ then $\|P(x_n - y_n)\| \to 0$. But $P(x_n - y_n) = x_n$ and $\{x_n\}$ does not converge to 0. Conversely, suppose $\inf\{\|x - y\| : x \in M, y \in N, \|x\| = 1 = \|y\|\} > 0$. Suppose $x_n \in M, y_n \in N$ for all n and $x_n + y_n \to z$. Let $a_n = \|x_n\|, b_n = \|y_n\|, u_n = \frac{1}{a_n} x_n, v_n = \frac{1}{b_n} y_n$. We have $a_n u_n + b_n v_n \to z$. Suppose $\alpha_n \equiv \|(a_n, b_n)\| \to \infty$. Then $t_n u_n + s_n v_n \to 0$ where $t_n = \frac{a_n}{\alpha_n}$ and $s_n = \frac{b_n}{\alpha_n}$. Since u_n and v_n are unit vectors we get $||t_n| - |s_n|| \leq \|t_n u_n + s_n v_n\| \to 0$. Since $t_n^2 + s_n^2 = 1$ it follows that, through a subsequence, $t_n \to t, s_n \to s$ with $s = \pm t (= \pm \frac{1}{\sqrt{2}})$. Now $\|t u_n + s v_n\| \leq \|t_n u_n + s_n v_n\| + |t_n - t| + |s_n - s| \to 0$ (all these limits are along a subsequence). But then $\|u_n \pm v_n\| \to 0$ and the hypothesis shows that $\|u_n \pm v_n\| \geq \delta$. Conclusion: α_n must be bounded. Going to a subsequence we may suppose $a_n \to a$ and $b_n \to b$ for some a, b. Since $a_n u_n + b_n v_n \to z$ we get $a u_n + b v_n \to z$. If $a = 0$ then $z = \lim(x_n + y_n) = \lim y_n \in N \subseteq M + N$. Similarly if $b = 0$ then $z \in M \subseteq M + N$. So assume that a and b are non-zero. If $\{u_n\}$ is Cauchy (hence convergent) so is $\{v_n\}$ and we get $z \in M + N$. If not there exists $\varepsilon > 0$ and $\{n_j\}, \{m_j\} \uparrow \infty$ such that $\left\|u_{n_j} - u_{m_j}\right\| > \varepsilon$ for all j. Let $w_j = \frac{u_{n_j} - u_{m_j}}{\left\|u_{n_j} - u_{m_j}\right\|}, z_j = \frac{v_{n_j} - v_{m_j}}{\left\|v_{n_j} - v m_j\right\|}, r_j = \left\|u_{n_j} - u_{m_j}\right\|, \rho_j = \left\|v_{n_j} - v_{m_j}\right\|$. Then $a r_j w_j + b \rho_j z_j \to 0$. Since $\{r_j\}$ and $\{\rho_j\}$ have convergent subsequences, say with limits r and ρ) we get $a r w_j + b \rho z_j \to 0$. But then $||ar| - |b\rho|| \leq \|a r w_j + b \rho z_j\| \to 0$ so $ar = \pm b\rho$. This gives $w_j \pm z_j \to 0$ which contradicts the hypothesis unless $ar = b\rho = 0$ which implies $r = \rho = 0$. But $r_j \geq \varepsilon$ so $r \geq \varepsilon$. This finishes the proof.

Problem 480

Show that $\lambda I - T$ is invertible if $\lambda \notin \{0, 1\}$ and $T^2 = T$; compute its inverse explicitly.

[T may be a bounded operator, in which case the inverse is also a bounded operator) or just a linear idempotent map]

The inverse is $\frac{1}{\lambda}I+\frac{1}{\lambda(\lambda-1)}T$. To guess this write $(\lambda I-T)^{-1}=\lambda^{-1}\sum\limits_{n=0}^{\infty}\frac{1}{\lambda^n}T^n=\lambda^{-1}(I+\sum\limits_{n=1}^{\infty}\frac{1}{\lambda^n}T)$.

Problem 481

Let X be a normed linear space. Show that X is a complete if and only if the intersection of any decreasing sequence of closed balls is non-empty.

Remark: the proof below can be adopted to show that a metric space (X,d) is complete if and only if the intersection of any decreasing sequence of closed balls with radii converging to 0 is non-empty. [Cantor's Intersection Theorem gives one part. See also problem 490].

Suppose the intersection of any decreasing sequence of closed balls is non-empty. Let $\{x_n\}$ be Cauchy. Let $\left\|x_{n_k}-x_{n_{k+1}}\right\|<\frac{1}{2^k}$ with $n_k\uparrow$ and $r_k=\frac{1}{2^{k-1}}$. Then the closed balls with centers at x_{n_k}, radius r_k are decreasing and if x is in their intersection then $x_{n_k}\to x$ which implies $x_n\to x$. Hence X is complete. Conversely let X be complete and let $\{\bar{B}(x_n,r_n)\}$ be a decreasing sequence of closed balls. Claim: $\{x_n\}$ is Cauchy. We have $x_{n+1}+\frac{r_{n+1}}{\|x_{n+1}-x_n\|}(x_{n+1}-x_n)\in\bar{B}(x_{n+1},r_{n+1})\subseteq\bar{B}(x_n,r_n)$ so $\left\|x_{n+1}+\frac{r_{n+1}}{\|x_{n+1}-x_n\|}(x_{n+1}-x_n)-x_n\right\|\le r_n$ which says $\|x_{n+1}-x_n\|+r_{n+1}\le r_n$ i.e. $\|x_{n+1}-x_n\|\le r_n-r_{n+1}$. This implies that $\{r_n\}$ is decreasing (which is also obvious from the fact that the diameters of $\bar{B}(x_n,r_n)$ are decreasing). Let $r_n\downarrow r$. Iteration of $\|x_{n+1}-x_n\|\le r_n-r_{n+1}$ yields $\|x_{n+m}-x_n\|\le r_n-r_{n+m}\to 0$ as $n,m\to\infty$ so the Cauchy sequence $\{x_n\}$ has a limit x. Now, letting $m\to\infty$ in $\|x_{n+m}-x_n\|\le r_n-r_{n+m}$ we get $\|x-x_n\|\le r_n-r\le r_n$ so the intersection of the balls $\bar{B}(x_n,r_n)$ contains x. Note that the intersection of $\bar{B}(x_n,r_n),n=1,2,...$ is precisely $\bar{B}(x,r)$.

Problem 482

Given distinct real numbers $\{x_1,x_2,...,x_k\}\subseteq[0,1]$, $\varepsilon>0$ and a continuous function f on $[0,1]$ show that there is a polynomial p such that $\|f-p\|_\infty<\varepsilon$ and $p(x_i)=f(x_i)$ for $1\le i\le k$.

Let $q_i(x)=\dfrac{\prod\limits_{j\ne i}(x-x_j)}{\prod\limits_{j\ne i}(x_i-x_j)}$. Then q_i is a polynomial, $q_i(x_i)=1$ and $q_i(x_j)=0$ if $j\ne i$. Let $\rho=\sum\limits_{i=1}^{k}a_iq_i$. Then $\rho(x_i)=a_i,1\le i\le k$. Also $\|\rho\|_\infty\le M\max\{|a_i|:1\le i\le k\}$ where M depends only on $\{x_1,x_2,...,x_k\}$. Now choose a polynomial p_0 such that $\|f-p_0\|_\infty<\varepsilon/(1+M)$. Let $a_i=f(x_i)-p_0(x_i),1\le i\le k$. Let ρ correspond to this choice of $a_i's$. Let $p=p_0+\rho$. Then $p(x_i)=p_0(x_i)+a_i=$

$f(x_i), 1 \le i \le k$. Also $\|p - f\|_\infty \le \|p_0 - f\|_\infty + \|\rho\|_\infty < \varepsilon/(1+M) + M\varepsilon/(1+M) = \varepsilon$ since $|a_i| < \varepsilon/(1+M)$ for each i.

Problem 483

Show that $\sum_{n=1}^{\infty} a_n e^{inx}$ is the Fourier series of a C^∞ periodic function if and only if $\{n^k \hat{f}(n)\}$ is bounded for each k.

Recall that $\left|\hat{f}(n)\right| \le \frac{C}{n}$, $n \ne 0$ if f is C^1. Also $\hat{f}^{(k)}(n) = (in)^k \hat{f}(n)$. Thus $\left|n^k \hat{f}(n)\right| = \left|\hat{f}^{(k)}(n)\right|$ and $\{n^k \hat{f}(n)\}$ is bounded if f is C^∞. Conversely suppose $k \ge 2$ is fixed and $\left|n^k \hat{f}(n)\right| \le C_k$ with C_k independent of n. Then repeated differentiation of the Fourier series of f shows that $f \in C^{(k-2)}$.

Problem 484

Prove that $\sup\{\int_0^y \frac{\sin x}{x} dx : y > 0\} = \int_0^\pi \frac{\sin x}{x} dx \ne \int_0^\infty \frac{\sin x}{x} dx$.

Let $a_n = \int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x} dx$. Note that $\sin x$ is alternately positive and negative in $(0,\pi), (\pi, 2\pi), ...$ Hence $a_n > 0$ if n is odd, < 0 if n is even. Also Then $a_{n+1} = \int_{(n-1)\pi}^{n\pi} \frac{\sin(x+\pi)}{x+\pi} dx = -\int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x+\pi} dx$. It follows that $|a_{n+1}| < |a_n|$ for all n. [Indeed $|a_n| = \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{x} dx$ and $|a_{n+1}| = \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{x+\pi} dx$]. Let $s_n = \sum_{j=1}^{n} a_j$. If n is odd then $s_n = b_1 + (b_3 - b_2) + ... + (b_n - b_{n-1})$ where $b_n = |a_n|$. It follows that $s_n \le b_1$. If n is even then $s_n \le s_{n-1} \le b_1$. Thus $s_n \le |a_1| = a_1$ for all n. This means $\int_0^{n\pi} \frac{\sin x}{x} dx \le \int_0^\pi \frac{\sin x}{x} dx$ for each n. Also the fact that $a_n's$ alternate in sign and $a_1 > 0$ implies that $s_n > 0$ for all n. If $(n-1)\pi \le y \le n\pi$ with n then $\int_0^y \frac{\sin x}{x} dx = \int_0^{(n-1)\pi} \frac{\sin x}{x} dx + \int_{(n-1)\pi}^{y} \frac{\sin x}{x} dx \le \int_0^{(n-1)\pi} \frac{\sin x}{x} dx \le \int_0^{\pi} \frac{\sin x}{x} dx$ if n is even.

If n is odd then $\int_0^y \frac{\sin x}{x} dx = \int_0^{(n-1)\pi} \frac{\sin x}{x} dx + \int_{(n-1)\pi}^{y} \frac{\sin x}{x} dx \le \int_0^{n\pi} \frac{\sin x}{x} dx \le \int_0^{\pi} \frac{\sin x}{x} dx$.

In the first case $\int\limits_0^y \frac{\sin x}{x} dx \geq \int\limits_0^{(n-1)\pi} \frac{\sin x}{x} dx + \int\limits_{(n-1)\pi}^{n\pi} \frac{\sin x}{x} dx = s_n > 0$ and in the second case $\int\limits_0^y \frac{\sin x}{x} dx = \int\limits_0^{(n-1)\pi} \frac{\sin x}{x} dx + \int\limits_{(n-1)\pi}^{y} \frac{\sin x}{x} dx \geq \int\limits_0^{(n-1)\pi} \frac{\sin x}{x} dx = s_{n-1} > 0$. We have proved that $\left| \int\limits_0^y \frac{\sin x}{x} dx \right| \leq \int\limits_0^{\pi} \frac{\sin x}{x} dx$ for every $y > 0$ proving that $\sup\{ \int\limits_0^y \frac{\sin x}{x} dx : y > 0\} = \int\limits_0^{\pi} \frac{\sin x}{x} dx$. Note: we have proved that $\int\limits_0^y \frac{\sin x}{x} dx > 0$ for all $y > 0$. If $\int\limits_0^{\pi} \frac{\sin x}{x} dx = \int\limits_0^{\infty} \frac{\sin x}{x} dx$ then $a_2 + a_3 + ... = 0$. This is impossible because $a_2 + a_3 < 0, a_4 + a_5 < 0, ...$

Problem 485

Let C be a closed convex set in a normed linear space X and let $x \in C$. Show that $\{y \in X : x + ty \in C \ \forall t > 0\}$ is independent of $x \in C$.

Let $x_1, x_2 \in C$ and y be such that $x_1 + ty \in C \ \forall t > 0$. We have $x_2 + ty = \lim\limits_{n \to \infty} [(1 - \frac{1}{n})x_2 + \frac{1}{n}\{x_1 + nty\}]$. Since $x_1 + nty \in C$ we get $(1 - \frac{1}{n})x_2 + \frac{1}{n}\{x_1 + nty\} \in C \ \forall n$ so $x_2 + ty \in C$.

Problem 486

Let $C_i, i = 1, 2, 3, 4$ be convex sets in $\mathbb{R}^2$. If any three of these have non-empty intersection show that all four of them have non-empty intersection. Prove that if any two of three convex sets in $\mathbb{R}^2$ have non-empty intersection it does not follow that all three of them have non-empty intersection. Generalize to $\mathbb{R}^n$.

For the counter-example look at the coordinate axes and the line $\{(x.y) : x + y = 1\}$. Now let $x_i \in \bigcap\limits_{j \neq i} C_j$. Let $y_i = x_i - x_4, 1 \leq i \leq 3$. There three vectors are linearly dependent. Let $\sum\limits_{i=1}^{3} a_i y_i = 0$ with not all of a_1, a_2, a_3 equal to 0. We have $\sum\limits_{i=1}^{4} b_i x_i = 0$ where $b_i = a_i$ for $1 \leq i \leq 3$ and $b_4 = -(a_1 + a_2 + a_3)$. Thus $\sum\limits_{i=1}^{4} b_i = 0$ and not all the $b_i's$ are 0. Partition $\{1, 2, 3, 4\}$ into two sets I and J

by $I = \{i \in \{1,2,3,4\} : b_i \geq 0\}$ and $J = \{i \in \{1,2,3,4\} : b_i < 0\}$. We have $\sum_{i \in I} |b_i| \, x_i = \sum_{i \in J} |b_i| \, x_i$. Also $\sum_{i \in I} |b_i| = \sum_{i \in J} |b_i| = c$ (say). Writing c_i for $\frac{|b_i|}{c}$ we get $\sum_{i \in I} |c_i| \, x_i = \sum_{i \in J} |c_i| \, x_i$. The left side belongs to $\bigcap_{j \in J} C_j$ and the right side to $\bigcap_{j \in I} C_j$. Hence $\sum_{i \in I} |c_i| \, x_i = \sum_{i \in J} |c_i| \, x_i$ belongs to every C_i. Generalization to $\mathbb{R}^n$ is straightforward.

Problem 487

Give two closed convex sets in $\mathbb{R}^2$ whose sum is not closed.

$C_1 = \{(x,y) : x > 0 \text{ and } y \geq \frac{1}{x}\}, C_2 = \{(x,y) : x < 0 \text{ and } y \geq -\frac{1}{x}\}$. Since $(n, \frac{2}{n}) + (-n, \frac{2}{n}) \to (0,0)$ we see that $C_1 + C_2$ is not closed.

Problem 488

If $f : \mathbb{R} \to \mathbb{R}$ is convex and bounded above show that f is a constant.

Let $x_1 < x_2$. If $f(x_1) < f(x_2)$ then, for n sufficiently large, we have $x_2 = ax_1 + (1-a)n$ where $a = \frac{n-x_2}{n-x_1}$ and $f(x_2) \leq af(x_1) + (1-a)f(n) \leq af(x_1) + (1-a)\sup\{f(t) : t \in \mathbb{R}\}$. Since $\frac{f(x_2)-af(x_1)}{1-a} \to \infty$ as $n \to \infty$ we get $\sup\{f(t) : t \in \mathbb{R}\} = \infty$. If $f(x_2) < f(x_1)$ then $x_1 = a(-n) + (1-a)x_2$ where $a = \frac{x_2-x_1}{x_2+n}$ and $f(x_1) \leq af(-n) + (1-a)f(x_2) \leq a\sup\{f(t) : t \in \mathbb{R}\} + (1-a)f(x_2)$. Since $\frac{f(x_1)-(1-a)f(x_2)}{a} \to \infty$ as $n \to \infty$ we $\sup\{f(t) : t \in \mathbb{R}\} = \infty$ again. It follows that if $\sup\{f(t) : t \in \mathbb{R}\} < \infty$ then $f(x_1) = f(x_2)$ whenever $x_1 < x_2$.

Problem 489

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be convex in the first variable and continuous in the second variable. Show that f is continuous.

Fix $(a,b) \in \mathbb{R}^2$. Let $\varepsilon > 0$. Choose $r > 0$ such that $|f(x,b) - f(a,b)| < \varepsilon$ if $|x-a| \leq r$. Choose $s > 0$ such that $|f(a-r,y) - f(a-r,b)| < \epsilon, |f(a,y) - f(a,b)| < \epsilon$ and $|f(a+r,y) - f(a+r,b)| < \epsilon$ if $|y-b| \leq s$. Now let $|x-a| \leq r$ and $|y-b| \leq s$. Write (x,y) as $\alpha(a-r,y) + (1-\alpha)(a+r,y)$ where $\alpha = \frac{a+r-x}{2r}$. We have $f(x,y) \leq \alpha f(a-r,y) + (1-\alpha)f(a+r,y) \leq \alpha[\varepsilon + f(a+r,b)] + (1-\alpha)[\varepsilon + f(a+r,b)]$

$\leq \varepsilon + \alpha[f(a,b)+\varepsilon] + (1-\alpha)[f(a,b)+\varepsilon] = f(a,b) + 2\varepsilon$. On the other hand, if $a-r \leq x \leq a$ then $(a,y) = \beta(a+r,y) + (1-\beta)(x,y)$ where $\beta = \frac{a-x}{a+r-x}$ and $f(a,b) - \varepsilon < f(a,y) \leq \beta f(a+r,y) + (1-\beta)f(x,y) < \beta[f(a+r,b)+\varepsilon] + (1-\beta)f(x,y)$

$< \beta[f(a,b)+2\varepsilon] + (1-\beta)f(x,y)$ so $f(x,y) > \frac{1}{1-\beta}[(1-\beta)f(a,b) - 2\varepsilon\beta] = f(a,b) - \frac{2\varepsilon\beta}{1-\beta}$. Noting that $\beta \to 0$ as $x \to a$ we see that $\liminf f(x,y) \geq f(a,b)$.

For $a \leq x \leq a+r$ we write $(a,y) = \gamma(a-r,y) + (1-\gamma)(x,y)$ and use a similar argument.

Problem 490

Show that there exists a complete metric space (X,d) and a decreasing sequence of closed balls in it whose intersection is empty.

[Solution from stackexchange.com]. Let $X = \mathbb{N}, d(n,m) = 1 + \left|\frac{1}{2^n} - \frac{1}{2^m}\right|$ if $n \neq m$, 0 if $n = m$. Let B_n be the closed ball with center $n+1$ and radius $1 + \frac{1}{2^{n+1}}$. Indeed, $d(n+1,m) \leq 1 + \frac{1}{2^{n+1}}$ iff $n+1 = m$ or $0 < \left|\frac{1}{2^{n+1}} - \frac{1}{2^m}\right| \leq \frac{1}{2^{n+1}}$ iff $m \geq n$. Since any Cauchy sequence is a constant, the space is complete.

Problem 491

Let $\{x_i\}$ be linearly independent in a vector space V over $K (= \mathbb{R}$ or $\mathbb{C})$. Show that there is an inner product on V which makes $\{x_i\}$ an orthonormal set.

Let $A \equiv \{x_i\} \cup \{y_j\}$ be a Hamel basis of V. Let $X = L^2(A)$, the space of all functions $f : A \to K$ with $\sum\limits_{a \in A} |f(a)|^2 < \infty$. X is a Hilbert space under the inner product $< f, g >= \sum\limits_{a \in A} f(a)[g(a)]^-$. Define a linear map $\phi : V \to X$ by defining $\phi(a) = \delta_a$ for $a \in A$ and extending ϕ by linearity to all of V. Here $\delta_a(b) = 1$ if $b = a$ and 0 otherwise. Define $< x, y >=< \phi(x), \phi(y) >$ for $x, y \in V$. Then $< x_i, x_j >=< \phi(x_i), \phi(x_j) >=< \delta_{x_i}, \delta_{x_j} >= 1$ if $i = j$, 0 if $i \neq j$.

Problem 492

If A is an uncountable subset of $\mathbb{R}$ show that there exists $a \in \mathbb{R}$ such that $A \cap (-\infty, a)$ and $A \cap (a, \infty)$ are both uncountable.

Suppose not. Then $\mathbb{R} = E \cup F$ where $E = \{x \in \mathbb{R} : A \cap (-\infty, x)$ is at most countable$\}$ and $F = \{x \in \mathbb{R} : A \cap (x, \infty)$ is at most countable$\}$. Since A is uncountable the sets E and F are disjoint. If we show that E and F are closed we get a contradiction to the fact that $\mathbb{R}$ is connected. Suppose $\{x_n\} \subseteq E$ and $x_n \to x$. Then $A \cap (-\infty, x) \subseteq \bigcup\limits_{n=1}^{\infty} A \cap (-\infty, x_n)$ so $x \in E$. Hence E is closed. Similarly, F is closed.

Problem 493

Let $A_{r,s}$ denote the annulus $\{z \in \mathbb{C} : r < |z| < s\}$. Show that $A_{0,1}$ is not conformally equivalent to $A_{1,2}$.

If there is a conformal equivalence f of $A_{0,1}$ onto $A_{1,2}$ then f extends to a holomorphic function on U. Let $f(0) = a$. By open mapping theorem $a \in A_{1,2}$. Let $b \in A_{0,1}$ with $a = f(b)$. Let V_1 and V_2 be disjoint neighbourhoods of 0 and b respectively. Then $f(V_1) \cap f(V_2)$ is a nonempty open set. Let $x \in f(V_1) \cap f(V_2) \backslash \{a\}$. Then $x = f(v_1) = f(v_2)$ for some $v_1 \in V_1, v_2 \in V_2$. Also $v_1 \neq 0$ since $x \neq a$. Thus $v_1, v_2 \in A_{0,1}$ and $v_1 \neq v_2$. This contradicts the fact that f is injective.

Problem 494

Show that a countable subset A of a real normed linear space X is connected if and only if it is a singleton.

Note that $x^* \in X^*$ implies $x^*(A)$ is a connected countable subset of $\mathbb{R}$, hence a singleton set. say $\{c\}$. If $a_1, a_2 \in A$ then $x^*(a_1), x^*(a_2) \in \{c\}$ so $x^*(a_1) = x^*(a_2)$. This holds for all x^* so $a_1 = a_2$.

Problem 495

Let A be an $n \times n$ complex matrix. Prove that if $\lim a^n A^n$ exists and is non-zero then $\lim a^n \lambda^n$ exists for every eigen value λ. If A has n distinct real eigen values show that $a = \frac{1}{\lambda}$ for some eigen value λ.

There exists a matrix S such that $B \equiv SAS^{-1}$ is upper triangular. Clearly, $\lim a^n A^n$ exists and is non-zero iff $\lim a^n B^n$ exists and is non-zero. If this is true then $\lim a^n \lambda^n$ exists each eigen value λ because the diagonal elements of B are the eigen values of A. If A has n distinct real eigen values then there exists a basis consisting of eigen vectors and $\lim a^n A^n$ exists and is non-zero iff $\lim a^n \lambda^n$ exists and is non-zero each eigen value λ. This implies that $a = \frac{1}{\lambda}$ where $\lambda = \max\{\mu : \mu \text{ is an eigen value of } A\}$.

Problem 496

If $f \in C([0,1])$ show that $\lim\limits_{n\to\infty} (n+1) \int\limits_0^1 x^n f(x)dx = f(1)$.

If $f(x) = \sum\limits_{k=0}^{m} a_k x^k$ then $(n+1)\int\limits_0^1 x^n f(x)dx = (n+1)\sum\limits_{k=0}^{m} a_k \frac{1}{k+n} \to \sum\limits_{k=0}^{m} a_k = f(1)$ as $n \to \infty$. For the general case let $\varepsilon > 0$ and let p be a polynomial such that $|f(x) - p(x)| < \varepsilon$ for all x. Then $\left|(n+1)\int\limits_0^1 x^n f(x)dx - (n+1)\int\limits_0^1 x^n p(x)dx\right| \leq \varepsilon(n+1)\int\limits_0^1 x^n dx = \varepsilon$ and $|f(1) - p(1)| < \varepsilon$. Second proof: Let $|f(x) - f(1)| < \varepsilon$

for $1-\delta \leq x \leq 1$. Then $(n+1)\int\limits_0^{1-\delta} x^n |f(x)|\, dx \leq (n+1)(1-\delta)^n \|f\|_\infty \to 0$ and $(n+1)\int\limits_{1-\delta}^{1} x^n f(x)dx - f(1) = (n+1)\int\limits_{1-\delta}^{1} x^n[f(x)-f(1)]dx - f(1)(1-\delta)^n$. Now $\left|(n+1)\int\limits_{1-\delta}^{1} x^n[f(x)-f(1)]dx\right| \leq \varepsilon(n+1)\int\limits_{1-\delta}^{1} x^n dx \leq \varepsilon$ and $f(1)(1-\delta)^n \to 0$ as $n \to \infty$.

Problem 497

Show that any linear map T on $\mathbb{R}^n$ with $n > 1$ has a two dimensional invariant subspace.

If T has an eigen value λ with $\operatorname{Im}\lambda \neq 0$ there there exists $x \in \mathbb{C}^n\backslash\{0\}$ such that $Tx = \lambda x$. Let y and z be the vectors obtained by taking the real and imaginary parts of the components of x. Then $Ty = T(\frac{x+\bar{x}}{2})\frac{Tx+(Tx)^-}{2} = \frac{\lambda x+\bar{\lambda}\bar{x}}{2} = \frac{\lambda(y+iz)+\bar{\lambda}(y-iz)}{2} \in span\{y,z\}$ since $\lambda+\bar{\lambda}$ and $i(\lambda-\bar{\lambda})$ are real. Similarly, $Tz \in spam\{y,z\}$. Hence $span\{y,z\}$ is invariant. Suppose now that all eigen values of T are real. Let $Tx = \lambda x$ where $x \neq 0$. Define $S : \mathbb{R}^n/[x] \to \mathbb{R}^n/[x]$ by $S(y+[x]) = Tx+[x]$. [Here $[x]$ denotes the span of $\{x\}$]. Then S is a well defined linear map on the $n-1$ dimensional space $\mathbb{R}^n/[x]$ and it has an invariant one dimensional subspace spanned by a vector $z+[x]$. [Argue as before if there is a complex eigen value. If there is a real eigen value then there is a real eigen vector]. Now $span\{x,z\}$ is invariant for T.

Problem 498

Show that there is a continuous monotonic function $f:(0,1) \to \mathbb{R}$ such that $\int\limits_0^1 |f(x)|\, dx = \infty$ and $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n-1} \frac{1}{n} f(\frac{k}{n})$ exists (and is finite).

Let $f(x) = \frac{1}{x} - \frac{1}{1-x}$. We have $\sum\limits_{k=1}^{n-1} \frac{1}{n} f(\frac{k}{n}) = \sum\limits_{k=1}^{n-1} \frac{1}{n}\{\frac{n}{k} - \frac{n}{n-k}\} = \sum\limits_{k=1}^{n-1}\{\frac{1}{k} - \frac{1}{n-k}\} = \sum\limits_{k=1}^{n-1} \frac{1}{k} - \sum\limits_{k=1}^{n-1} \frac{1}{k} = 0$.

Problem 499

Let c be a complex number such that convergence of $\{a_0+a_1+...+a_{n-1}+ca_n\}$ implies that of $\{a_0+a_1+...+a_{n-1}+a_n\}$. Show that either $c=0$ or $\operatorname{Re} c > \frac{1}{2}$. Prove that the converse is also true.

Let $f(s) = \sum_{n=0}^{\infty} a_n s^n$, $g(s) = \sum_{n=0}^{\infty} b_n s^n$ and $h(s) = \sum_{n=0}^{\infty} c_n s^n$ where $b_0 = ca_0, b_n = a_0 + a_1 + ... + a_{n-1} + ca_n$ and $c_n = a_0 + a_1 + ... + a_{n-1} + a_n$ for $|s| \leq 1$. Then $(1-s)h(s) = c_0 + \sum_{n=1}^{\infty}(c_n - c_{n-1})s^n = a_0 + \sum_{n=1}^{\infty} a_n s^n = f(s)$. Also $(1-s)g(s) = f(s) - (1-c)(1-s)f(s)$. [This can be seen by comparing coefficients of s^n]. Hence $h(s) = \frac{f(s)}{1-s} = \frac{g(s)}{1-(1-c)(1-s)} = \frac{g(s)}{c+s-cs}$. This equation holds for $|s| < 1$ and hence for $|s| < \min\{1, \frac{|c|}{|1-c|}\}$;assuming $c \neq 0, c \neq 1$ we see that $c_n =$ coefficient of s^n in $\frac{g(s)}{c}(1 + \frac{1-c}{c}s\ )^{-1} \equiv \frac{g(s)}{c}\sum_{k=0}^{\infty}(-\frac{1-c}{c})^k s^k$ which is $\frac{(-1)^n}{c}\{b_0(\frac{1-c}{c})^n + b_1(\frac{1-c}{c})^{n-1} + ... + b_n\}$. We have proved that $c_n = \frac{(-1)^n}{c}\{b_0(\frac{1-c}{c})^n + b_1(\frac{1-c}{c})^{n-1} + ... + b_n\}$ for all n. Now let $\{\alpha_n\}$ be any convergent sequence of complex numbers. Then we can choose $\{a_n\}$ such that $b_n = \alpha_n$ for all n. It follows by hypothesis that $\{c_n\}$ converges. It follows that $\{\frac{1}{c}\{\alpha_0(\frac{c-1}{c})^n + \alpha_1(\frac{c-1}{c})^{n-1} + ... + \alpha_n\}\}$ converges whenever $\{\alpha_n\}$ does. A standard argument using Uniform Boundedness Principle shows that $\sum_{k=0}^{\infty}\left|\frac{c-1}{c}\right|^k < \infty$ which means $\left|\frac{c-1}{c}\right| < 1$. Thus $1 + |c|^2 - 2\operatorname{Re} c < |c|^2$ so $\operatorname{Re} c > \frac{1}{2}$. This proves the direct part. For the converse let $\operatorname{Re} c > \frac{1}{2}$. Assume $c \neq 0$. Then $\left|\frac{1-c}{c}\right| < 1$ and so $\{c_n\} = \{\frac{1}{c}\{b_0(\frac{c-1}{c})^n + b_1(\frac{c-1}{c})^{n-1} + ... + b_n\}\}$ converges whenever $\{b_n\}$ converges. The case $c = 0$ is trivial.

[We have used the following theorem above: suppose $\{\sum_{k=0}^{n} a_{k,n}b_k\}$ converges whenever $\{b_n\}$ does. Then $\sup_n \sum_{k=0}^{n}|a_{k,n}| < \infty$. For a proof let X be the Banach space of all convergent sequences of complex numbers with the supremum norm. Define $T_m : X \to \mathbb{C}$ by $T_m\{b_n\} = \sum_{k=0}^{m} a_{k,m}b_k$ for $m = 1, 2, ...$ Note that $T_m\{b_n\} = \sum_{k=0}^{m}|a_{k,m}|$ when $b_k = \frac{|a_{k,m}|}{a_{k,m}}$ if $k \leq m$ and $a_{k,m} \neq 0$, 1 if $k \leq m$ and $a_{k,m} = 0, b_k = 0$ for $k > m$. Since $\|\{b_n\}\| = 1$ we see that $\|T_m\| \geq \sum_{k=0}^{m}|a_{k,m}|$. It suffices, therefore to show that $\sup_m \|T_m\| < \infty$. However the sequence $\{T_m\}$ of bounded operators on X converges at each point $\{b_n\}$ of X and the result follows by Uniform Boundedness Principle].

Problem 500

Let $p(z) = a_0 + a_1 z + ... + a_n z^n$ with each $a_j > 0$. Let α and β be the minimum and maximum of the numbers $\frac{a_{j+1}}{a_j}, 0 \le j \le n-1$. Show that every zero z of p satisfies the inequalities $\frac{1}{\beta} \le |z| \le \frac{1}{\alpha}$.

First consider a polynomial $q(z) = b_0 + b_1 z + ... + b_n z^n$ with $b_0 > b_1 > ... > b_n > 0$. If $q(z) = 0$ and $|z| \le 1$ then $0 = |(1-z)(b_0 + b_1 z + ... + b_n z^n)| = \left| b_0 + (b_1 - b_0)z + (b_2 - b_1)z^2 + ... + (b_n - b_{n-1})z^n - b_n z^{n+1} \right|$
$\ge b_0 - \{(b_0 - b_1) + (b_1 - b_2) + ... + (b_{n-1} - b_n) + b_n\} = 0$ and this forces z to be 1. Since $q(1) \ne 0$ it follows there is no zero of q in the closed unit disk. Now let $t > 0$. Then $p(\frac{z}{t}) = b_0 + b_1 z + ... + b_n z^n$ with $b_j = \frac{a_j}{t^j}$. If $t > \max\limits_j \frac{a_{j+1}}{a_j} \equiv \beta$ then $b_0 > b_1 > ... > b_n > 0$. Thus $p(\frac{z}{t}) = 0$ implies $|z| > 1$. In other words every zero ζ of p satisfies $|\zeta| > \frac{1}{t}$. This is true whenever $t > \beta$. Letting $t \to \beta$ we see that $|\zeta| \ge \frac{1}{\beta}$ for every zero ζ of p. Similarly considering $p(\frac{t}{z})$ with $0 < t < \alpha$ we see that $|\zeta| \le \frac{1}{a}$ for every zero ζ of p.

Problem 501

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable with compact support. Show that $f(x+iy) = -\frac{1}{\pi} \iint\limits_{\mathbb{R}^2} \frac{(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f(\zeta)}{2(\zeta - z)} d\xi d\eta$ where $\zeta = \xi + i\eta, \xi, \eta$ real.

Let $\phi(r,\theta) = f(z + re^{i\theta})$ and consider $\int\limits_{\varepsilon}^{\infty} \int_0^{2\pi} \{\frac{\partial}{dr} + \frac{i}{r}\frac{\partial}{\partial\theta}\}\phi(r,\theta) d\theta dr$. As $\varepsilon \to 0$ this converges to $2 \iint\limits_{\mathbb{R}^2} \frac{(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f(\zeta)}{2(\zeta - z)} d\xi d\eta$. By periodicity we get $\int_0^{2\pi} \frac{i}{r}\frac{\partial\phi}{\partial\theta} d\theta = 0$. Hence $-\frac{1}{\pi} \iint\limits_{\mathbb{R}^2} \frac{(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f(\zeta)}{2(\zeta - z)} d\xi d\eta$

$= -\frac{1}{2\pi} \lim \int\limits_{\varepsilon}^{\infty} \int_0^{2\pi} \frac{\partial}{dr}\phi(r,\theta) d\theta dr = -\frac{1}{2\pi} \lim \int_0^{2\pi} \{0 - \phi(\varepsilon,\theta)\} d\theta$. But $\phi(\varepsilon,\theta) \to f(z)$ uniformly in θ so we get $-\frac{1}{\pi} \iint\limits_{\mathbb{R}^2} \frac{(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f(\zeta)}{2(\zeta - z)} d\xi d\eta = f(z)$
.

Problem 502

Let A be a subset of $\mathbb{R}^n$ such that every continuous real valued function on it extends to a continuous function on $\bar{A}$. Show that A is closed, but need not be compact.

The second part follows by taking $A = \mathbb{N}$ in $\mathbb{R}$. For the first part suppose $\{a_n\} \subseteq A, a_n \to a$ and $a \notin A$. Let $f(a_n) = n, n = 1, 2, \ldots$. Since $\{a_n\}$ has no lmit points in A Tietze Extension Theorem shows that there is a continuous function on A which extends f. This extended functions obviously does not extend to a continuous function on $\bar{A}$.

Problem 503

Let A be a subset of $\mathbb{C}$ such that every continuous real valued function on A can be approximated uniformly on A by polynomials in x and y. Show that A is compact.

Remark: the converse of this follows immediately from Stone -Weirstrass Theorem. See more remarks at the end of the solution below.

Let $f : A \to \mathbb{R}$ be continuous. Let $a \in \partial A$. Choose $\{a_n\} \subseteq A$ such that $a_n \to a$. Let $\varepsilon > 0$. Let p be a polynomial such that $|f(x) - p(x)| < \varepsilon \ \forall x \in A$. Since $\{p(a_n)\}$ is convergent, hence Cauchy there exists n_0 such that $|p(a_n) - p(a_m)| < \varepsilon \ \forall n, m \geq n_0$. This gives $|f(a_n) - f(a_m)| < 3\varepsilon \ \forall n, m \geq n_0$. Thus $\lim\limits_{n\to\infty} f(a_n)$ exists. Call this limit $f(a)$. If $\{b_n\}$ is another sequence in A converging to a then $\{f(a_1), f(b_1), f(a_2), f(b_2), \ldots\}$ is Cauchy, hence convergent. This proves that $f(a)$ does not depend on the choice of $\{a_n\}$. We have extended f to $\bar{A}$. We claim that f is continuous on $\bar{A}$. Let $\{a_n\}$ and $a \in \partial A$ and $a_n \to a$. Since p is continuous at a there exists $\delta > 0$ such that $|p(x) - p(a)| < \varepsilon$ if $|x - a| < \delta$. We can find b_n, b in A such that $|f(a_n) - f(b_n)| < \varepsilon, |f(a) - f(b)| < \varepsilon, |a_n - b_n| < \delta/2$ and $|b - a| < \delta/2$. Now $|f(a_n) - f(a)| \leq |f(a_n) - f(b_n)| + |f(b_n) - p(b_n)| + |p(b_n) - p(b)| + |p(b) - f(b)| + |f(b) - f(a)| < 5\varepsilon$ for n so large that $|b_n - b| < \frac{\delta}{2} + |a_n - a| + |a - b| < \delta$. This proves that f is continuous. We have proved that any continuous function on A extends to a continuous function on $\bar{A}$. This implies that A is closed: suppose $\{a_n\} \subseteq A, a_n \to a \notin A$. Then $\{a_n\}$ is closed in A and Tietze Extension Theorem shows that there exists a continuous function f on A such that $f(a_n) = n \ \forall n$. This function does not extend to a continuous function on $\bar{A}$. Thus, A is necessarily closed. Suppose A is unbounded. Let f be a bounded continuous function on A. There exists a polynomials $p_n, n = 1, 2 \ldots$ such that $|f(x) - p_n(x)| < \frac{1}{n} \ \forall x \in A$. It follows that p_n is bounded on the unbounded set A and hence constant. Hence $f = \lim p_n$ is a constant too. Thus every continuous bounded function from A to $\mathbb{R}$ is a constant. Let $\{a_n\} \subseteq A$ and $|a_n| \to \infty$. Applying Tietze Theorem again we see that there exists a bounded continuous function g on A (in fact on $\mathbb{C}$) such that $g(a_n) = \frac{1}{n} \ \forall n$. This function is not constant.

Remarks: the proof shows that every continuous function on A extends to a continuous function on $\bar{A}$ if and only if A is closed. Approximating f by polynomials in $x + iy$ is a different story altogether. $\frac{1}{z}$ is continuous on T but it cannot be approximated uniformly by polynomials in z. Mergelyan"s Theorem says that if A is compact and $\mathbb{C} \backslash A$ is connected the every continuous function on A can be approximated uniformly on A by polynomials in z.

Problem 504

Show that every sequence of real numbers has a monotone subsequence.

Let $\{a_n\} \subseteq \mathbb{R}$ and $E = \{k : a_n \leq a_k \ \forall n > k\}$. [Points of E are called the *peaks* of $\{a_n\}$]. If E is empty or finite there exists k_0 such that $k > k_0$ implies $a_n > a_k$ for at least one $n > k$. We can choose an increasing subsequence of $\{a_n\}$ inductively in this case. Suppose E is an infinite set. Let $n_1 < n_2 < ...$ with each $n_j \in E$. Then $a_{n_{j+1}} \leq a_{n_j}$ because $n_j \in E$ and $n_{j+1} > n_j$. Hence $\{a_{n_j}\}$ is a decreasing subsequence of $\{a_n\}$.

Problem 505

If A is a convex set in $\mathbb{R}^n$ show that A is closed if and only if $A \cap L$ is closed for every straight line L in $\mathbb{R}^n$.

Using the results in my notes convexity.tex this problem is quite easy. Here is a sketch: if A has an interior point (which may be assumed to be the origin) then $x \in \partial A \Rightarrow$ the line L joining 0 and x intersects A in a line segment with x as an end point. Since $A \cap L$ is closed it follows that $x \in A$. Hence A is closed. If A has empty interior then there is a lower dimensional space in which the previous argument works.

Problem 506.

Let X be a real normed linear space with $\dim X \geq 2$. Let $x \neq 0$ and $\alpha > 0$. Show that there exists $y \in X$ with $\|y\| = \alpha$ and $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Define $f : S \equiv \{y \in X : \|y\| = \alpha\} \to \mathbb{R}$ by $f(y) = \|x+y\|^2 - \|x\|^2 - \|y\|^2$.

f is continuous and S is connected. Indeed, if y_1 and $y_2 \in S, y_1 \neq y_2, y_1 \neq -y_2$ then $t \to \alpha \frac{ty_2+(1-t)y_1}{\|ty_2+(1-t)y_1\|}$ is a path in S connecting y_1 and y_2. If $y_1 = -y_2$ we can connect y_1 and y_2 to $\alpha \frac{y_1+y_2}{\|y_1+y_2\|}$. It follows now that the range of f is an interval. Now $f(\alpha \frac{x}{\|x\|}) = (\|x\| + \alpha)^2 - \|x\|^2 - \alpha^2 = 2\alpha \|x\| > 0$ whereas $f(-\alpha \frac{x}{\|x\|}) = (\|x\| - \alpha)^2 - \|x\|^2 - \alpha^2 = -2\alpha \|x\| < 0$. We conclude that f must vanish at some point $y \in S$.

Problem 507 [Stability if linear independence]

Let $\{x_1, x_2, \ldots, x_N\}$ be linearly independent elements of a normed linear space X. Show that there exists $\varepsilon > 0$ such that $\|y_i - x_i\| < \varepsilon$ for $i = 1, 2, .., N$ implies $\{y_1, y_2, \ldots, y_N\}$ is linearly independent.

Suppose this is false. Fix $\varepsilon > 0$. Choose vectors $y_1, y_2, \ldots, y_N$ and scalars $c_1, c_2, \ldots, c_N$ not all 0 such that $\sum\limits_{i=1}^{N} c_i y_i = 0$ and $\|y_i - x_i\| < \varepsilon$ for $i = 1, 2, .., N$.

Then $\left\|\sum_{i=1}^{N} c_i x_i\right\| = \left\|\sum_{i=1}^{N} c_i(x_i - y_i)\right\| < \varepsilon \sum_{i=1}^{N} |c_i|$. Denoting $\frac{c_i}{\sum_{i=1}^{N}|c_i|}$ by d_i we get $\left\|\sum_{i=1}^{N} d_i x_i\right\| < \varepsilon$. Note that $\sum_{i=1}^{N} |d_i| = 1$. If we let $\varepsilon \to 0$ and use compactness of $\{(d_1, d_2, \ldots, d_N) \in \mathbb{R}^N : \sum_{i=1}^{N} |d_i| = 1\}$ we get $\sum_{i=1}^{N} t_i x_i = 0$ for some $(t_1, t_2, \ldots, t_N)$ with $\sum_{i=1}^{N} |t_i| = 1$. This contradicts the linear independence of $\{x_1, x_2, \ldots, x_N\}$.

Problem 508

Let $x_1, x_2, \ldots, x_N$ be unit vectors in a real normed linear space X such that $\left\|\sum_{i=1}^{N} c_i x_i\right\| \leq M \max_{1 \leq i \leq N} |c_i|$ for all $c_1, c_2, \ldots, c_N$. Show that $\left\|\sum_{i=1}^{N} c_i x_i\right\| \geq (2 - M) \max_{1 \leq i \leq N} |c_i|$for all $c_1, c_2, \ldots, c_N$.

Define $T : (\mathbb{R}^N, \|\|_\infty) \to X$ by $T(c_1, c_2, \ldots, c_N) = \sum_{i=1}^{N} c_i x_i$. The hypothesis says that $\|T\| \leq M$. Hence $\|T^*\| \leq M$. It is easy to see that $T^* : X^* \to (\mathbb{R}^N, \|\|_1)$ is given by $T^* x^* = (x^*(x_1), x^*(x_2), \ldots, x^*(x_N))$. Let $\max_{1 \leq i \leq N} |c_i| = |c_j|$. Choose x^* such that $\|x^*\| = 1$ and $x^*(x_j) = 1$. Then $\left\|\sum_{i=1}^{N} c_i x_i\right\| \geq \left|x^*(\sum_{i=1}^{N} c_i x_i)\right| \geq |c_j| - \max_{1 \leq i \leq N} |c_i| \sum_{i \neq j} |x^*(x_i)|$. We claim that $\sum_{i \neq j} |x^*(x_i)| \leq M - 1$. Since $T^*(x^*) = (x^*(x_1), x^*(x_2), \ldots, x^*(x_N))$ we have $\sum_{i=1}^{N} |x^*(x_i)| \leq M$. Combined with the fact that $x^*(x_j) = 1$ we get $\sum_{i \neq j} |x^*(x_i)| \leq M - 1$ as claimed. Now $\left\|\sum_{i=1}^{N} c_i x_i\right\| \geq |c_j| - \max_{1 \leq i \leq N} |c_i| \sum_{i \neq j} |x^*(x_i)| \geq |c_j| - (M - 1) \max_{1 \leq i \leq N} |c_i| \geq (2 - M) \max_{1 \leq i \leq N} |c_i|$ since $\max_{1 \leq i \leq N} |c_i| = |c_1|$.

Problem 509

If $\{a_n\}, \{b_n\} \subseteq \mathbb{R}$ and $a_n \cos nt + b_n \sin nt \to 0$ for all t in (a, b) for some $a < b$ show that $a_n \to 0$ and $b_n \to 0$.

[See also Problem 510]

We have to show that $a_n^2+b_n^2 \to 0$. Suppose not. Then $\frac{a_n \cos nt + b_n \sin nt}{\sqrt{a_n^2+b_n^2}} \to 0$ along a subsequence and Dominated Convergence Theorem shows $\int\limits_a^b \frac{|a_n \cos nt + b_n \sin nt|^2}{a_n^2+b_n^2} dt \to 0$ along a subsequence. Now $\int\limits_a^b \frac{|a_n \cos nt + b_n \sin nt|^2}{a_n^2+b_n^2} dt = \frac{a_n^2}{a_n^2+b_n^2}(\frac{b-a}{2} + o(1)) + \frac{b_n^2}{a_n^2+b_n^2}(\frac{b-a}{2} + o(1)) + \frac{2a_n b_n}{a_n^2+b_n^2} o(1)$. It follows that $\frac{a_n^2}{a_n^2+b_n^2}\frac{b-a}{2} + \frac{b_n^2}{a_n^2+b_n^2}\frac{b-a}{2} \to 0$ along a subsequence, which is absurd.

Problem 510

If X is a real normed linear space, $\{x_n\} \cup \{y_n\} \subseteq X$ and $x_n \cos nt + y_n \sin nt \to 0$ for all t in some interval (a,b) show that $x_n \to 0$ and $y_n \to 0$.

Remark: weak convergence of $\{x_n\}$ and $\{y_n\}$ is trivial from previous problem. What is asserted here is norm convergence.

We compute $\int\limits_a^b |x^*(x_n \cos nt + y_n \sin nt)|^2 \, dt$. We get $(x^*(x_n))^2(\frac{b-a}{2} + o(1)) + (x^*(y_n))^2(\frac{b-a}{2} + o(1)) + 2x^*(x_n)x^*(y_n)o(1)$. Note that $\{x_n\}$ and $\{y_n\}$ are norm bounded (because they converge to 0 weakly). Since the above integral tends to 0 (by DCT), $(x^*(x_n))^2 + (x^*(y_n))^2 \to 0$. Now observe that $\int\limits_a^b |x^*(x_n \cos nt + y_n \sin nt)|^2 \, dt \leq \int\limits_a^b \|x_n \cos nt + y_n \sin nt\|^2 \, dt$ if $\|x^*\| \leq 1$ and $\int\limits_a^b \|x_n \cos nt + y_n \sin nt\|^2 \, dt \to 0$ by DCT so $(x^*(x_n))^2 + (x^*(y_n))^2 \to 0$ uniformly for $\|x^*\| \leq 1$. This implies that $|x^*(x_n)| \to 0$ uniformly for $\|x^*\| \leq 1$ and $|x^*(y_n)| \to 0$ uniformly for $\|x^*\| \leq 1$ and this completes the proof.

Problem 511

Let $X = l_1$ and $M = \{\{a_n\} \in X : 0 = a_1 = a_3 = a_5 = \dots\}$. Show that any non-zero continuous linear functional on M has infinitely many norm preserving extensions to X.

Let f be a non-zero continuous linear functional on M. By Hahn Banach Theorem and the fact that $(l_1)^* = l_\infty$ there exists a non-zero element $\{c_n\}$ of l_∞ such that $f(\{a_n\}) = \sum\limits_{i=1}^{\infty} a_i c_i = \sum\limits_{i=1}^{\infty} a_{2i} c_{2i}$. Also $\|f\| = \sup\limits_n |c_n|$. However $|f(\{a_n\})| = \left| \sum\limits_{i=1}^{\infty} a_{2i} c_{2i} \right| \leq \|\{a_n\}\| \sup\limits_n |c_{2n}|$ so $\sup\limits_n |c_n| \leq \sup\limits_n |c_{2n}|$ which implies $\|f\| = \sup_n |c_n| = \sup_n |c_{2n}|$. Let $g(\{a_n\}) = \sum_{i=1}^{\infty} a_{2i}c_{2i} + \sum_{i=1}^{\infty} \beta_{2i+1}a_{2i+1}$ where $\{\beta_{2i+1}\}$ is an arbitrary bounded sequence with $\sup_n |\beta_{2n+1}| = \sup_n |c_n|$. Of course, distinct $\{\beta_n\}'s$ give distinct linear functionals g. We claim that each one of these is a norm preserving extension of f. Of course, g extends f and $\|g\| \leq \max\{\sup_n |c_{2n}|, \sup_n |\beta_{2n+1}|\} = \sup_n |c_{2n}| = \|f\|$. This implies $\|g\| = \|f\|$.

Problem 512

Let $f : (0,1) \to \mathbb{R}$ be continuous and $\int_0^1 x^n f(x)dx = 0$ for $n = 0, 1, 2, \ldots$. Show that $f(x) = 0 \ \forall x \in (0,1)$.

Let $g(x) = \int_0^x f(t)dt$. Then $\int_0^1 x^n g(x)dx = \int_0^1 x^n \int_0^x f(t)dtdx = \frac{x^{n+1}}{n+1}\int_0^x f(t)dt|_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}f(x)dx = \frac{1}{n+1}\int_0^1 f(t)dt = 0$ and $g \in C([01])$. Hence $g \equiv 0$ which implies $f(x) = g'(x) = 0$ for $0 < x < 1$.

Problem 513

Let f be a continuously differentiable function on $[0,1]$ with $f(0) = f(1) = 0$. Show that $\|f\|_1 \leq \frac{1}{4}\|f'\|_\infty$.

Let $f_1(x) = f(x), f_2(x) = -f(x), f_3(x) = f(1-x)$ and $f_4(x) = -f(1-x)$. Then $\|f_j\|_\infty = \|f\|_\infty, 1 \leq j \leq 4$. Claim: $f_j(x) \leq (\|f'\|_\infty)x$ for $0 < x < 1, 1 \leq j \leq 4$. To see this observe that $f_j(x) = \int_0^x f_j'(t)dt \leq (\|f'\|_\infty)x$. It follows that $|f(x)| \leq (\|f'\|_\infty)x$ and $|f(1-x)| \leq (\|f'\|_\infty)x$ so $|f(x)| \leq (\|f'\|_\infty)(1-x)$. Hence $\|f\|_1 \leq \|f'\|_\infty \int_0^1 \min\{x, 1-x\}dx = \frac{1}{4}\|f'\|_\infty$.

Problem 514

a) If $f \in C([0,1])$ show that $\sum_{k=0}^{n} \frac{1}{n} f(\frac{k}{n}) \to \int_0^1 f(x)dx$

b) If $f \in C^1([0,1])$ show that $\sum_{k=0}^{n} f(\frac{k}{n}) - n\int_0^1 f(x)dx \to \frac{f(1)-f(0)}{2}$

c) Give an example to show the conclusion of b) fails for some $f \in C([0,1])$.

Counter-example first: let $\phi(x) = \left\{ \begin{array}{c} x \text{ if } 0 \le x \le \frac{1}{2} \\ 1-x \text{ if } \frac{1}{2} \le x \le 1 \end{array} \right.$ and $f(x) = \sum\limits_{k=0}^{\infty} \frac{\phi(2^k x)}{2^k}$. We have $\sum\limits_{k=0}^{2^n} f(\frac{k}{2^n}) = \sum\limits_{k=0}^{2^n}\sum\limits_{j=0}^{\infty} \frac{\phi(2^j \frac{k}{2^n})}{2^j} = \sum\limits_{j=0}^{n-1}\sum\limits_{k=0}^{2^n} \phi(k2^{j-n})\frac{1}{2^j} = \sum\limits_{j=0}^{n-1} 2^{j-n+1} \sum\limits_{k=0}^{2^{n-j-1}} k =$ $2^{n-1} - \frac{1}{2}$ so $\sum\limits_{k=0}^{2^n} f(\frac{k}{n}) - 2^n \int\limits_0^1 f(x)dx = -\frac{1}{2}$ whereas $\frac{f(1)-f(0)}{2} = 0$.

a) is trivial.

b) $\sum\limits_{k=0}^{n} f(\frac{k}{n}) - n\int\limits_0^1 f(x)dx = \sum\limits_{k=0}^{n}\{f(\frac{k}{n}) - n \int\limits_{(k-1)/n}^{k/n} f(x)dx\} = n\sum\limits_{k=0}^{n} \int\limits_{(k-1)/n}^{k/n} \{f(\frac{k}{n}) -$ $f(x)\}dx$

$= n\sum\limits_{k=0}^{n} \int\limits_{(k-1)/n}^{k/n} (\frac{k}{n} - x)f'(\xi_{k,x})dx = n\sum\limits_{k=0}^{n} \int\limits_{(k-1)/n}^{k/n} (\frac{k}{n} - x)\{f'(\xi_{k,x}) - f'(\frac{k}{n})\}dx +$ $n\sum\limits_{k=0}^{n} \int\limits_{(k-1)/n}^{k/n} (\frac{k}{n} - x)f'(\frac{k}{n})dx$. Using uniform continuity of f' and the fact that $\left| n\sum\limits_{k=0}^{n} \int\limits_{(k-1)/n}^{k/n} (\frac{k}{n} - x)dx \right| \le 1$ we see that the first term above can be made arbitrarily small by choosing n sufficiently large. Note also that $n\sum\limits_{k=0}^{n} \int\limits_{(k-1)/n}^{k/n} (\frac{k}{n} -$ $x)f'(\frac{k}{n})dx = n\sum\limits_{k=0}^{n} f'(\frac{k}{n}) \int\limits_{(k-1)/n}^{k/n} (\frac{k}{n} - x)dx = n\sum\limits_{k=0}^{n} f'(\frac{k}{n})\frac{1}{2n^2} \to \frac{1}{2}\int\limits_0^1 f'(y)dy =$ $\frac{f(1)-f(0)}{2}$.

Problem 515

Show that $f : [a,b] \to \mathbb{R}$ is absolutely continuous if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite disjoint collection of intervals $\{(a_i, b_i) :$ $1 \le i \le k\}$ with $\sum\limits_{i=1}^{k}(b_i - a_i) < \delta$ we have $\left|\sum\limits_{i=1}^{k}\{f(b_i) - f(a_i)\}\right| < \varepsilon$.

Consider those intervals (a_j, b_j) for which $f(b_i) - f(a_i) \geq 0$. Since the sum of the lengths of these intervals is less than δ we get $\sum |f(b_j) - f(a_j)| < \varepsilon$ where the sum if taken over these intervals. Similar argument holds for intervals with $f(b_i) - f(a_i) < 0$. Hence $\sum_{j=1}^{k} |f(b_j) - f(a_j)| < 2\varepsilon$ proving that f is absolutely continuous.

Problem 516

Show that $f : [a, b] \to \mathbb{R}$ is Lipschitz if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite collection of intervals $\{(a_i, b_i) : 1 \leq i \leq k\}$ with $\sum_{i=1}^{k} (b_i - a_i) < \delta$ we have $\sum_{i=1}^{k} |f(b_i) - f(a_i)| < \varepsilon$.

Note that the intervals are not necessarily disjoint. Let $\varepsilon = 1$ and choose δ correspondingly. Let $a < b$. Consider the points $t_0 = a, t_1 = a + \frac{\delta}{2n}, \ldots, t_m = a + \frac{m\delta}{2n}$ and $t_{m+1} = b$ where m is defined by the inequalities $t_m \leq b < a + \frac{(m+1)\delta}{2n}$. Considering the collection $(t_{j-1}, t_j), (t_{j-1}, t_j), \ldots, (t_{j-1}, t_j)$ (the interval (t_{j-1}, t_j) repeated n times) we get $n\,|f(t_j) - f(t_{j-1})| < 1$. Hence $|f(b) - f(a)| \leq \sum |f(t_j) - f(t_{j-1})| < \frac{m}{n}$. Note that $m \leq \frac{2n(b-a)}{\delta} < m + 1$ so $|f(b) - f(a)| \leq \frac{2(b-a)}{\delta}$.

Problem 517

Suppose $f : [0, 1] \to \mathbb{R}$ is continuous. If $f'(x) = 0$ for all irrational numbers x show that f is a constant. What if $f'(x) = 0$ a.e.?

We prove that if $f' = 0$ except on a countable set A then f is a constant. Suppose f is not a constant. Then there exists $x_0 > 0$ such that $\alpha \equiv f(x_0) - f(0) \neq 0$. Replacing f by $-f$ if necessary we may assume that $\alpha > 0$. Let $0 < t < \alpha$. For $0 < \beta < \frac{a-t}{x_0}$ define $g_\beta(x) = f(x) - f(0) - \beta x$. Note that $g_\beta(x_0) = \alpha - \beta x_0 > t$. Let $\xi_\beta = \sup\{x \in (0, x_0) : g_\beta(x) \leq t\}$. Clearly, $0 < \xi_\beta < x_0$. We claim that $g_\beta(\xi_\beta) = t$. Assuming this for the moment consider the map $\beta \in (0, \min\{\frac{a-t}{x_0}, 1\}) \to \xi_\beta$. Call this map ϕ. If $\phi(\beta_1) = \phi(\beta_2)$ then $\xi_{\beta_1} = \xi_{\beta_2}$ and $g_{\beta_1}(\xi_{\beta_1}) - g_{\beta_2}(\xi_{\beta_1}) = g_{\beta_1}(\xi_{\beta_1}) - g_{\beta_2}(\xi_{\beta_2})$
$= t - t = 0$ which gives $f(\beta_1) - f(0) - \beta_1^2 = f(\beta_1) - f(0) - \beta_2\beta_1$ which implies $\beta_1 = \beta_2$. We get the desired contradiction by showing that ξ_β must be in the countable set A. Otherwise $f'(\xi_\beta) = 0$. However $0 < h < x_0 - \xi_\beta$ implies $g_\beta(\xi_\beta + h) > t = g_\beta(\xi_\beta)$ (by the claim) so $\frac{g_\beta(\xi_\beta + h) - g_\beta(\xi_\beta)}{h} > 0$ whereas $0 = f'(\xi_\beta) = \beta + g'_\beta(\xi_\beta)$ so $g'_\beta(\xi_\beta) < 0$, a contradiction. It remains now to prove the claim. If $g_\beta(\xi_\beta) < t$ then $g_\beta(z) = t$ for some $z \in (\xi_\beta, y)$ where $y \in (0, x_0)$ and $g_\beta(y) > t$. [y exists because $g_\beta(x_0) > t$ and g_β is continuous. z exists by intermediate value property]. But this contradicts the definition of ξ_β. Since $g_\beta(\xi_\beta) \leq t$ [by continuity of g_β and the definition of ξ_β] the claim is proved.

Problem 518

Suppose $f : [0,1] \to \mathbb{R}$ is continuous. If $f'(x)$ exists and is non-negative for all irrational x show that f is monotonically increasing. What if $f'(x)$ exists and is non-negative a.e.?

We prove that if $f'(x)$ exists and is non-negative except on a countable set A then f is monotonically increasing.

Since $\frac{d}{dx}\{f(x) + \varepsilon x\} > 0$ outside A and f is be increasing if $f(x) + \varepsilon x$ is increasing for each $\varepsilon > 0$ we may assume that $f'(x)$ exists and is strictly positive outside A. We assume that $f(b) < f(a)$ with $a < b$ and arrive at a contradiction. Since $(f(b), f(a))$ is uncountable it is not contained in the countable set $f(A)$. Let $u \in (f(b), f(a)) \backslash f(A)$. Let $\xi = \sup\{x \in [a,b] : f(x) \geq u\}$. Then $\xi < b$. [The set defining ξ is non-empty because $f \geq u$ near a. Since $f < u$ near b we must have $\xi < b$]. By continuity we have $f(\xi) = u$. For any $\delta > 0$, $\sup\{f(x) : x \in [a, \xi) \cap (\xi - \delta, \xi)\} \geq u$ by definition of ξ. Letting $\delta \to 0$ we get a sequence $\{x_n\} \uparrow \xi$ such that $f(x_n) > f(\xi)\ \forall n$ (because $u > f(\xi)$). But then $\frac{f(\xi) - f(x_n)}{\xi - x_n} < 0$ which implies that $\xi \in A$. Hence $u = f(\xi) \in f(A)$, a contradiction.

Problem 519

Let $f \in C([0,1]), 0 < c < 1$ and $\lim\limits_{h \in Q, h \to 0} \frac{f(c+h) - f(c)}{h} = l (\in \mathbb{R})$. Show that $f'(c) = l$.

Let $\varepsilon, \delta > 0$. By uniform continuity there exists $h_\delta \in \mathbb{Q}$ such that $|f(c+\delta) - f(c+h_\delta)| < \varepsilon\delta$ and $|h_\delta - \delta| < \delta^2$. We have $\left|\frac{f(c+\delta)-f(c)}{\delta} - l\right| \leq \left|\frac{f(c+h_\delta)-f(c)}{\delta} - l\right| + \left|\frac{f(c+\delta)-f(c+h_\delta)}{\delta}\right| < \left|\frac{f(c+h_\delta)-f(c)}{\delta} - l\right| + \varepsilon < 2\varepsilon$ if δ is sufficiently small.

Problem 520

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $|f(x) - f(y)| \geq a\,|x - y|\ \forall x, y$ where $a > 0$. What can you say about the range of f?

f is necessarily surjective: its range is an interval. Since f is injective and continuous, it is monotonic. We may suppose f is increasing in which case $f(na) \geq na$ by induction so f is unbounded above. Similarly the inequality $f(x) \leq f(y) - a(y - x)$ for $x < y$ shows that f is unbounded below. Hence f is a homeomorphism of $\mathbb{R}$.

Problem 521

Prove or disprove: any continuous one-to-one function from $\mathbb{Q}$ into itself is monotone.

[Recall that any continuous one-to-one function from $\mathbb{R}$ into itself is monotone].

False: let $f(x) = \begin{cases} \frac{x}{4} \text{ if } x < \sqrt{2} \\ \frac{1}{x} \text{ if } \sqrt{2} < x < 2\sqrt{2} \\ 3x \text{ if } x > 2\sqrt{2} \end{cases}$. f is decreasing in $(\sqrt{2}, 2\sqrt{2})$ and increasing in $(2\sqrt{2},\infty)$. It is clearly continuous and one-to-one on $\mathbb{Q}$.

Problem 522

Let p be a polynomial with real coefficients. If all the roots of p are real show that $pp'' \leq (p')^2$. Is this true even if p has non-real roots?

Let $p(x) = \prod_{k=1}^{n}(x-a_k)$ with $a_k's$ real. Then $p'(x) = p(x)\sum_{k=1}^{n} \frac{1}{x-a_k}$ and $p''(x) = p'(x)\sum_{k=1}^{n} \frac{1}{x-a_k} - p(x)\sum_{k=1}^{n} \frac{1}{(x-a_k)^2}$. Hence $p(x)p''(x) \leq p(x)p'(x)\sum_{k=1}^{n} \frac{1}{x-a_k} = (p'(x))^2$. For $p(x) = x^2+1$ we have $p(x)p''(x) = 2(x^2+1) > 4x^2 = (p'(x))^2$ when $|x| < 1$.

Problem 523

a) Let E be a measurable subset of $\mathbb{R}$ such that $E + \frac{1}{n} = E$ $\forall n \geq 1$. Show that $m(E) = 0$ or $m(E^c) = 0$.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable, $f(x+\alpha) = f(x)$ $\forall x, f(x+\beta) = f(x)$ $\forall x$ where α, β are non-zero real numbers with $\frac{\alpha}{\beta}$ irrational. Prove that f is a.e. constant. Give an example to show that f need not be a constant.

a) Let $m(E) > 0, a \in \mathbb{R}$ and $f(x) = m(E \cap [a,x])$ for $a \leq x < \infty$. If $a < x < y$ then $f(y+\frac{1}{n}) - f(x+\frac{1}{n}) = m(E \cap (x+\frac{1}{n}, y+\frac{1}{n}])$
$= m(\{E - \frac{1}{n}\} \cap (x,y]) = m(E \cap (x,y])$ and $f(y-\frac{1}{n}) - f(x-\frac{1}{n}) = m(E \cap (x-\frac{1}{n}, y-\frac{1}{n}])$
$= m(\{E+\frac{1}{n}\} \cap (x,y]) = m(E \cap (x,y])$. It follows that $f(y+\frac{1}{n}) - f(x+\frac{1}{n}) = f(y-\frac{1}{n}) - f(x-\frac{1}{n})$. Note that $|f(y) - f(x)| \leq |y-x|$ so f is absolutely continuous. Hence it is differentiable almost everywhere and using above equation we conclude that its derivative is a constant c a.e.. Since $\frac{m(x-\delta,x+\delta)}{2\delta} = \frac{f(x+\delta)-f(x-\delta)}{2\delta}$ and almost all points of E have metric density 1 we see that $c = 1$ Thus $f(y) - f(x) = \int_x^y f'(t)dt = y - x$. This gives $f(y) = f(a) + y - a$ $\forall x > a$. Thus $f(y) - f(x) = y - x$ or $m(E \cap (x,y]) = m((x,y])$ for $a < x < y$. This gives $m(E^c \cap (x,y]) = 0$ for $a < x < y$ which clearly implies that $m(E^c) = 0$.

b) Let $E = \{x : f(x) < a\}$. Then $E + t = E$ for t of the form $n\alpha + m\beta$ $(n, m \in \mathbb{Z})$. There is a sequence $\{t_j\}$ of numbers of this type decreasing to 0 since numbers of the form $n\alpha + m\beta$ $(n, m \in \mathbb{Z})$ form a dense subset of $\mathbb{R}$by Problem 87 above).

It is clear that the sequence $\{\frac{1}{n}\}$ in part a0 can be replaced by any sequence of positive numbers converging to 0. Since $E + t_j = E$ for each j it follows that $m(E) = 0$ or $m(E^c) = 0$. We have proved that for any real number a either $f < a$ a.e or $f \geq a$ a.e.. If $c = \sup\{a : m\{x : f(x) < a\} = 0\}$ it follows easily that $f = c$ a.e.. If f is the indicator function of $\{n\sqrt{2} + m\sqrt{3} : n, m \in \mathbb{Z}\}$ it follows that f has $\sqrt{2}$ and $\sqrt{3}$ as periods but f is not a constant.

Problem 524

Prove that any set of positive (Lebesgue) measure in $\mathbb{R}$ contains a non-mesurable set.

Let E be a bounded measurable set of positive measure. Let D be a countable infinite subset of E. Let H be the subgroup of $\mathbb{R}$ generated by D. [H consists of finite sums $\sum_{j=1}^{m} n_j d_j$ where m is a positive integer, $n_j's$ are integers and $d_j's$ belong to D]. Enumerate the distinct cosets of H as $\{H + t_i : i \in I\}$. Let $J = \{i \in I : (H + t_i) \cap E \neq \emptyset\}$. Pick an element x_j in $(H + t_j) \cap E$ for each $j \in J$. Let $L = \{x_j : j \in J\}$. Note that $E \subseteq \bigcup_{j \in J} (H + t_j)$. Since E is uncountable and H is countable it follows that J must be uncountable too. Let $S = H \cap (E - E)$. Then $D - D \subseteq S \subseteq H$ and S is countable. We claim that the sets $s + L$ $(s \in S)$ are disjoint. Suppose $s_1 \neq s_2$ and $(s_1 + L) \cap (s_2 + L) \neq \emptyset$. Then there exist $l_1, l_2 \in L$ such that $l_2 = s_1 + l_1 - s_2 \in l_1 + H$ and $l_1 \neq l_2$. This contradicts the fact that $H + t_i, i \in I$ are disjoint (so that $l_1 + H$ and $l_2 + H$ are disjoint since $H + x_j = t_j + H$ for each j and). This proves the claim. We claim that L is a non-measurable subset of E. Suppose L is measurable. If $m(L) > 0$ then $m(S + L) = \sum_{s \in S} m(s + L) = \infty$ and if $m(L) = 0$ then $m(S + L) = \sum_{s \in S} m(s + L) = 0$. We now prove that $m(S + L)$ can neither be 0 nor be ∞. Let us first prove that $E \subseteq S + L$. If $x \in E$ then $x \in (H + t_i) \cap E$ for some $i \in I$. This implies $i \in J$. Recalling that $x_i \in (H + t_i) \cap E$ we see that $x \in H + x_i$. Let $h = x - x_i$ so $h \in H$. Since $x - x_i \in E - E$ we get $h \in H \cap (E - E) = S$. Now $x = x_i + h \in L + S$. This proves that $E \subseteq S + L$. It follows that $m(S + L)$ cannot be 0. It cannot be ∞ either because $S + L$ is bounded: $S + L \subseteq H \cap (E - E) + E \subseteq E - E + E$ and E is bounded. The proof is complete.

Problem 525

Show that a continuous function $f : [a, b] \to \mathbb{R}$ maps Lebesgue measurable sets to Lebesgue measurable sets if and only if it maps Lebesgue null sets to Lebesgue null sets

Suppose f maps Lebesgue null sets to Lebesgue null sets. If E is measurable then $E = \bigcup\limits_n K_n \cup D$ with $K_n's$ compact and D null. It follows that $f(E) = \bigcup\limits_n f(K_n) \cup f(D)$ is measurable. Conversely suppose f maps Lebesgue measurable sets to Lebesgue measurable sets. Let E be a null set. If $m(f(E)) > 0$ then there is a non-measurable set $S \subseteq f(E)$. [See Problem 524 above]. Now $S = f(f^{-1}(S) \cap E), f^{-1}(S) \cap E$ is a null set whose image is not measurable.

Problem 526

Prove that $\limsup\limits_{n\to\infty} \cos nx = 1$ for every $x \in \mathbb{R}$.

If $\frac{x}{2\pi}$ is irrational then $\{n\frac{x}{2\pi} + m : n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Let $\eta > 0$ and choose $\delta > 0$ such that $|y| < \delta$ implies $\cos y > 1 - \eta$. There exists $n, m \in \mathbb{Z}$ such that $\left|n\frac{x}{2\pi} + m\right| < \frac{\delta}{2\pi}$, so $|nx + 2\pi m| < \delta$. We get $\cos nx = \cos|nx + 2\pi m| > 1 - \eta$ and we may (by changing m to $-m$ if necessary) that n is positive. If $\frac{x}{2\pi} = \frac{p}{q}$ with p, q integers then $\cos nx = 1$ whenever n is a multiple of q so $\limsup\limits_{n\to\infty} \cos nx = 1$.

Remark: the following more general result is true: if f is measurable function with period 1 then $\limsup\limits_{n\to\infty} f(nx) = ess.\sup$ of f on $[0,1]$ for almost all x. [M. Eidelheit, 1937]

Problem 527

For any measurable function $f : [0,\infty) \to \mathbb{C}$ show that $\int\limits_0^\infty \frac{|f(x)|^2}{1+x^2} dx \leq (\frac{1}{2} + \pi)\sup\limits_\Delta \frac{1}{\Delta} \int\limits_0^\Delta |f(t)|^2 \, dt$.

Let $g(x) = \int\limits_0^x |f(t)|^2 \, dt$. Then $\int\limits_0^\Delta \frac{|f(x)|^2}{1+x^2} dx = g(x)\frac{1}{1+x^2}|_0^\Delta + \int\limits_0^\Delta g(x)\frac{2x}{(1+x^2)^2} dx = \frac{g(\Delta)}{1+\Delta^2} + \int\limits_0^\Delta \frac{g(x)}{x}\frac{2x^2}{(1+x^2)^2} dx$. Hence

$\int\limits_0^\Delta \frac{|f(x)|^2}{1+x^2} dx \leq \frac{g(\Delta)}{1+\Delta^2} + \int\limits_0^\Delta \frac{g(x)}{x}\frac{2(1+x^2)}{(1+x^2)^2} dx = \frac{g(\Delta)}{1+\Delta^2} + 2\int\limits_0^\Delta \frac{g(x)}{x}\frac{1}{1+x^2} dx \leq \frac{g(\Delta)}{2\Delta} + (\sup \frac{g(\Delta)}{\Delta}) 2\tan^{-1}\Delta \leq (\frac{1}{2} + \pi)(\sup \frac{g(\Delta)}{\Delta})$.

Problem 528

Prove that the maps $f \to \{\hat{f}(n)\}$ from $L^1([0,2\pi])$ into c_0 [the spaces of sequences converging to 0 with sup. norm] and $f \to \hat{f}$ from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$ [where $C_0(\mathbb{R})$ is the space of continuous functions which vanish at $\pm\infty$ with the sup norm] do not have closed range.

If they have closed range they would have bounded inverses and there would be constants C_1, C_2 such that $\|f\|_1 \leq C_1 \sup\{\left|\hat{f}(n)\right| : n \in \mathbb{Z}\}, \|f\|_1 \leq C_1 \sup\{\left|\hat{f}(t)\right| : t \in \mathbb{R}\}$. To get a contradiction from the first inequality use the fact that $\|D_n\|_1 \to \infty$ as $n \to \infty$ where as $\left|\hat{D}_n(k)\right| \leq 1 \;\forall n, k$. For the second part let $g_n = I_{(-n,n)}$. Consider $f_n(t) \equiv \{g_n * g_1\}\hat{}\,(t) = \hat{g}_n(t)\hat{g}_1(t) = \frac{1}{2\pi}\frac{2\sin nt}{t}\frac{2\sin t}{t} = \frac{2}{\pi}\frac{\sin t \sin nt}{t^2}$. Since $f_n \in L^1$ for each n we have $\hat{f}_n = g_n * g_1$ and $\left|\hat{f}_n(t)\right| \leq \sqrt{\frac{2}{\pi}}$. It suffices to show that $\|f_n\|_1 \to \infty$ as $n \to \infty$. We have $\int \left|\frac{\sin t \sin nt}{t^2}\right| dt = n\int \left|\frac{\sin \frac{s}{n} \sin s}{s^2}\right| ds \geq n\int_{-\Delta}^{\Delta} \left|\frac{\sin \frac{s}{n} \sin s}{s^2}\right| ds \geq 2n\int_0^{\Delta} \frac{2}{\pi}\frac{s}{n}\left|\frac{\sin s}{s^2}\right| ds = \frac{4}{\pi}\int_0^{\Delta} \left|\frac{\sin s}{s}\right| ds$ provided $\frac{\Delta}{n} < \frac{\pi}{2}$. Since $\int_0^{\infty} \left|\frac{\sin s}{s}\right| ds = \infty$ it follows that $\|f_n\|_1 \to \infty$ as $n \to \infty$.

Problem 529

Let f be a right continuous function of bounded variation on $\mathbb{R}$ and μ be the real measure with $\mu(-\infty, x] = f(x) \;\forall x$. Show that $|\mu|(\mathbb{R}) = V_f$, the total variation of f.

It is obvious that $V_f \leq |\mu|(\mathbb{R})$. For the reverse inequality it suffices to show that $\sum_{j=1}^{N} |\mu(E_j)| \leq V_f$ for any finite disjoint collection $\{E_1, E_2, \ldots, E_N\}$ of Borel sets. By regularity of $|\mu|$ it suffices to prove this when the sets $E_1, E_2, \ldots, E_N$ are disjoint compact sets. In this case we can separate these compact sets by disjoint open sets $V_j, 1 \leq j \leq N$. By the basic approximation theorem we can approximate $\mu(E_j)$ by $\mu(F_j)$ where F_j is a finite disjoint union of left-open right-closed intervals lying in V_j. Thus the proof is reduced to the inequality $\sum_{j=1}^{N} |\mu(E_j)| \leq V_f$ when the sets are disjoint half-open intervals. This last fact is obviously true.

Problem 530

Let $f : [0, 2\pi] \to \mathbb{C}$ be a function of bounded variation. Show that $\left|\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx\right| \leq \frac{V_f}{|n|}$ for all $n \in \mathbb{Z}\backslash\{0\}$, V_f being the total variation of f.

Remark: periodicity of f is not required for this result.

Without loss of generality we may suppose f is right continuous and $f(0) = 0$. There exists a real measure μ such that $\mu[0,x] = f(x)\ \forall x$. There exists a a real measure ν such that $\frac{d\nu}{dm} = e^{-inx}$. Since $\nu(\mathbb{R}) = 0$ for $n \neq 0$we get $\int\limits_0^{2\pi} f(x)d\nu(x) = -\int\limits_0^{2\pi} \nu[0,x)d\mu(x)$. Hence $\left|\frac{1}{2\pi}\int\limits_0^{2\pi} f(x)e^{-inx}dx\right| \leq \frac{1}{2\pi}\left|\int\limits_0^{2\pi} \frac{e^{-inx}-1}{-in}d\mu(x)\right| \leq \frac{1}{\pi|n|}\left|\mu\right|(\mathbb{R}) = \frac{V_f}{\pi|n|}$ by Problem 529.

Problem 531

Give a proof of the uniqueness theorem and the basic approximation theorem of measure theory which does not use outer measures and the extension theorem.

Let μ be a positive measure on $(\Omega, \mathcal{F})$ and let $\mathcal{A}$ be an algebra that generates $\mathcal{F}$. Let $f \in L^2(\mu), f \geq 0$ and suppose f is orthogonal to the spaces $M = \{\sum\limits_{j=1}^{N} a_j I_{A_j} : N \geq 1, a_j' s \in \mathbb{C}, A_j' s \in \mathcal{A}\}$. Then $\int\limits_\Omega f d\mu = 0$ (f is integrable!) and $\{A \in \mathcal{F} : \int\limits_A f d\mu = 0\}$ is a sigma algebra containing $\mathcal{A}$. Hence it contained $\mathcal{F}$ and $f = 0$ a.e.. Thus every non-negative L^2 function belongs to the closure of M in $L^2(\mu)$. The same is true of every function in $L^2(\mu)$. If $f \in L^2(\mu)$ there exists $\{f_n\} \subseteq M$ such that $\|f_n - f\|_2 \to 0$. In particular if $\varepsilon > 0$, $A \in \mathcal{F}$ and $\mu(A) < \infty$ then there exists a simple function $\sum\limits_{j=1}^{N} a_j I_{A_j}$ $(N \geq 1, a_j' s \in \mathbb{C}, A_j' s \in \mathcal{A})$ such that $\int \left| I_A - \sum\limits_{j=1}^{N} a_j I_{A_j}\right| d\mu < \varepsilon$. It is easy to see that if $a_j \notin \{0,1\}$ then $\mu(A \Delta A_j) = 0$. Hence $\int |I_A - I_B|\, d\mu < \varepsilon$ for some $B \in \mathcal{A}$.

Problem 532

Show that no set of positive Lebesgue measure is a set of uniqueness for Fourier series, i.e. $m(E) > 0 \Rightarrow$ there exists $f \in L^1[0,2\pi]$ such that $\sum\limits_{n=-\infty}^{\infty} \hat{f}(n)e^{inx} = 0$ for every $x \in [0,2\pi]\backslash E$ but $\hat{f}(n) \neq 0$ for at least one n.

Let $f = I_K$ where K is a compact subset of E with $m(K) > 0$. By Riemann"s Localization Theorem the Fourier series of f and that of the zero function have the same sum (namely 0) at every point of K^c. It follows that the Fourier series of f has sum 0 at every point of E^c. However $\hat{f}(0) = \frac{1}{2\pi}\int\limits_0^{2\pi} I_K dm > 0$

Problem 533

Prove that a continuous function $\phi : (a,b) \to \mathbb{R}$ is convex if and only if $\limsup\limits_{h\downarrow 0} \frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{h^2} \geq 0\ \forall x \in (a,b)$. As a corollary a twice differentiable function if convex iff its second derivative is non-negative.

If ϕ is convex and f is its right-hand derivative then $\phi(x+h)+\phi(x-h)-2\phi(x) = \int\limits_x^{x+h} f(t)dt - \int\limits_{x-h}^{x} f(t)dt$

$= \int\limits_x^{x+h} f(t)dt - \int\limits_x^{x+h} f(t-h)dt \geq 0$ since $f(t)-f(t-h) \geq 0$. Now suppose $\limsup\limits_{h\downarrow 0} \frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{h^2} > 0$. Suppose $\phi(\alpha t+(1-\alpha)s) > \alpha\phi(t)+(1-\alpha)\phi(s)$ for some $a < t < s < b$ and some $\alpha \in (0,1)$. Let $\psi(x) = \phi(x) - \phi(t) - \frac{x-t}{s-t}\{\phi(s)-\phi(t)\}$. Then $\psi(t) = \psi(s) = 0$ and ψ is continuous. Also $\psi(\alpha t + (1-\alpha)s) > 0$. Hence there exists $c \in (t,s)$ such that $\psi(c) = \sup\{\psi(y) : t \leq y \leq s\}$. This implies $\frac{\psi(x+h)+\psi(x-h)-2\psi(x)}{h^2} \leq 0$ and this yields $\frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{h^2} \leq 0$ for all sufficiently small $h > 0$, contradicting the assumption that $\limsup\limits_{h\downarrow 0} \frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{h^2} > 0$. We have now proved that the condition $\limsup\limits_{h\downarrow 0} \frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{h^2} > 0$ implies convexity of ϕ. Now suppose $\limsup\limits_{h\downarrow 0} \frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{h^2} \geq 0$. If $\phi_1(x) = \phi(x) + \varepsilon x^2$ then $\limsup\limits_{h\downarrow 0} \frac{\phi_1(x+h)+\phi_1(x-h)-2\phi_1(x)}{h^2} \geq 2$ and hence ϕ_1 is convex for every $\varepsilon > 0$. Letting $\varepsilon \to 0$ we conclude that ϕ is convex.

Problem 534

If $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\hat{f} \geq 0$ show that $\hat{f} \in L^1(\mathbb{R})$ and $f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \hat{f}(t)e^{itx}dt$ a.e. Is this true if the hypothesis that $f \in L^\infty(\mathbb{R})$ is dropped?

The second part is just the inversion formula. To show that $\hat{f} \in L^1(\mathbb{R})$ let $d\mu(x) = \frac{dx}{\sqrt{2\pi}}$ and consider $\int \hat{f}(t)e^{-\frac{|t|}{n}}dt = \int\{\int f(x)e^{-itx}d\mu(x)\}e^{-\frac{|t|}{n}}dt = \int\{\int e^{-itx}e^{-\frac{|t|}{n}}dt\}f(x)dx = \int \frac{1}{\pi}\frac{n}{1+n^2x^2}f(x)dx$ so $\int \hat{f}(t)e^{-\frac{|t|}{n}}dt \leq \|f_\infty\|\ \forall n$. We have used Fubini, the fact that $\int e^{-itx}e^{-\frac{|t|}{n}}dt = \frac{1}{\pi}\frac{n}{1+n^2x^2}$ (which follows by two integrations by parts) and the fact that $\int \frac{1}{\pi}\frac{n}{1+n^2x^2}dx = 1$. Letting $n \to \infty$ we conclude that $\int \hat{f}(t)dt \leq \|f\|_\infty$. If $f \in L^1(\mathbb{R})$ and $\hat{f} \geq 0$ it does not follow that $\hat{f} \in L^1(\mathbb{R})$. In fact for any $f \in L^1(\mathbb{R})\backslash L^1(\mathbb{R})$ the function $g(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x-y)\bar{f}(-y)dy$ is in $L^1(\mathbb{R})$ and $\hat{g}(t) = \left|\hat{f}(t)\right|^2 \geq 0$. If $\hat{g} \in L^1(\mathbb{R})$ then $\hat{f} \in L^2(\mathbb{R})$ which implies $f \in L^2(\mathbb{R})$, a contradiction.

Problem 535

If $f \in L^1(\mathbb{R}), g \in L^2(\mathbb{R})$ and $\hat{f} = \hat{g}$ a.e. show that $f = g$ a.e.

Remark: an "elementary" proof is expected, one that does not use distribution theory; $\hat{g}$ is defined via Plancherel Theorem and $\hat{f}(t) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x)e^{-itx}dx$.

1. Let $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and consider the functions $f * \phi$ and $g * \phi$. Since $f * \phi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ it follows that these two functions are both in $L^2(\mathbb{R})$ and they have the same Fourier transform. Hence $f * \phi = g * \phi$ a.e. Now let $\phi_n(x) = \frac{\sqrt{n}}{\sqrt{2\pi}}e^{-nx^2/2}$. These functions form an approximate identity in $L^1(\mathbb{R})$ and hence $f * \phi_n \to f$ at Lebesgue points of f. Also $(g * \phi_n)\hat{} = \hat{g}\hat{\phi}_n \to \hat{g}$ in $L^2(\mathbb{R})$ because $\hat{\phi}_n(t) = e^{-x^2/2n} \to 1$ boundedly. It follows that $g * \phi_n \to g$ in $L^2(\mathbb{R})$ which yields a.e. convergence for a subsequence. Hence $f = g$ a.e.

Problem 536

Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function such that $f_y = f$ a.e. for every y, where $f_y(x) = f(x-y)$. Show that there is a constant c such that $f = c$ a.e..

This is a simple consequence of Fubini: $\int\int |f(x-y) - f(x)|\, dxdy = 0$ which implies $\int\int |f(x-y) - f(x)|\, dydx = 0$; hence there x_0 such that $\int |f(x_0-y) - f(x_0)|\, dy = 0$ or $\int |f(t) - f(x_0)|\, dt = 0$ so $f = f(x_0)$ a.e..

Problem 537 (Bump functions)

a) Show that there exists a C^∞ function on $\mathbb{R}$ which is 0 on $(-\infty, 0]$, 1 on $[1, \infty)$ and has its range $[0, 1]$.
b) Show that there exists a C^∞ function on $\mathbb{R}$ which is 0 on $(-\infty, -2] \cup [2, \infty)$, 1 on $[-1, 1]$ and has its range $[0, 1]$.

For a) take $f(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ 1 \text{ if } x \geq 1 \\ [1 + \frac{e^{1/t}}{e^{1/(1-t)}}]^{-1} \text{ if } 0 < x < 1 \end{cases}$.

For b) take $g(x) = \begin{cases} 0 \text{ if } x \leq -2 \text{ or } x \geq 2 \\ 1 \text{ if } -1 \leq x \leq 1 \\ f(x+2) \text{ if } -2 < x < -1 \\ f(x-2) \text{ if } 1 < x < 2 \end{cases}$ where f is the function in a).

Problem 538

Consider a function ψ in $L^2(\mathbb{R})$ such that $< \psi, \psi(2^n x - k) >= 0 \; \forall n \in \mathbb{N}, \forall k \in \mathbb{Z}$ (where $<,>$ is the inner product in $L^2(\mathbb{R})$). Show that such a function cannot be a C^∞ function with compact support unless $\psi \equiv 0$.

Remark: in the language of Multi Resolution Analysis of wavelets this says there is no smooth wavelet with compact support for a Multi Resolution Analysis.

Suppose $< \psi, \psi(2^n x - k) >= 0 \; \forall n \in \mathbb{N}, \forall k \in \mathbb{Z}$, ψ is C^∞ and has support in $[-\Delta.\Delta]$. Consider $\int \psi(a + \frac{x}{2^n})\bar{\psi}(x)dx$. If a is a dyadic rational then this integral is 0 for n sufficiently large because $2^n \int \psi(y)\bar{\psi}(2^n(y-a))dy = 0$ if $2^n a$ is an integer which is true for all n sufficiently large if a is a dyadic rational. Now $\psi(a + \frac{x}{2^n}) \to \psi(a)$ as $n \to \infty$ boundedly for $-\Delta \leq x \leq \Delta$. Hence $\psi(a) \int \bar{\psi} = \lim \int \psi(a + \frac{x}{2^n})\bar{\psi}(x)dx = 0$ whenever a is a dyadic rational. It follows that if ψ is not identically 0 then $\int \psi = 0$. We next show that $\int x\psi(x)dx = 0$. Let $f(x) = \int\limits_{-\infty}^{x} \bar{\psi}(t)dt = \int\limits_{-\Delta}^{x} \bar{\psi}(t)dt$. Then $0 = \int \psi(a + \frac{x}{2^n})\bar{\psi}(x)dx = \psi(a + \frac{x}{2^n})f(x)|_{-\infty}^{\infty} - \frac{1}{2^n} \int\limits_{-\Delta}^{\Delta} \psi'(a + \frac{x}{2^n})f(x)dx$. The fact that $\int \psi = 0$ implies that f vanishes outside $[-\Delta.\Delta]$. Also $\int\limits_{-\Delta}^{x} \psi'(a + \frac{x}{2^n})f(x)dx \to \psi'(a) \int f(x)dx$. If ψ' vanishes on dyadic rationals it vanishes everywhere and this makes ψ a constant function with compact support! It follows that if ψ is not identically 0 then $\int f = 0$. However $\int f = \int\limits_{-\Delta}^{\Delta} \int\limits_{-\Delta}^{x} \psi(t)dtdx = \int\limits_{-\Delta}^{\Delta} \int\limits_{t}^{\Delta} \psi(t)dxdt = \int\limits_{-\Delta}^{\Delta} (\Delta - t)\psi(t)dt = \Delta \int \psi - \int t\psi(t)dt$.

It follows that $\int t\psi(t)dt = 0$. If $g(x) = \int\limits_{-\infty}^{x} f(t)dt$ then $\int\limits_{-\Delta}^{\Delta} \psi'(a + \frac{x}{2^n})f(x)dx =$

$\psi'(a+\frac{x}{2^n})g(x)|_{-\infty}^{\infty} - \int\limits_{-\Delta}^{\Delta} \psi''(a+\frac{x}{2^n})g(x)dx$ so $\int\limits_{-\Delta}^{\Delta} \psi''(a+\frac{x}{2^n})g(x)dx = 0$ for n sufficiently large if a is a dyadic rational. Since the only polynomial (of degree 0 or 1) which has compact support is the zero function we conclude as before that $\int\limits_{-\Delta}^{\Delta} g(x)dx$. This yields $\int\limits_{-\Delta}^{\Delta} \frac{\Delta^2-t^2}{2}\psi(t)dt = 0$ so $\int\limits_{-\Delta}^{\Delta} t^2\psi(t)dt = 0$. An induction argument shows that $\int\limits_{-\Delta}^{\Delta} t^n\psi(t)dt = 0$ for every positive integer n. [The induction hypothesis is $\int t^n\psi(t)dt = 0$ and $\int\limits_{-\Delta}^{\Delta} \psi^{(n)}(a+\frac{x}{2^n})f_n(t)dt = 0$ for n sufficiently large whenever a is a dyadic rational where $f_{k+1}(x) = \int\limits_{-\Delta}^{\Delta} f_k(t)dt$ and $f_0 = \bar{\psi}$]. It follows by Weierstrass Theorem that $\int h\psi = 0$ for every $h \in C[-\Delta, \Delta]$. Hence, taking $h = \bar{\psi}$ we get $\psi \equiv 0$.

Problem 539

Let $f \in L^2(\mathbb{R})$. Show that the following two conditions are equivalent:

a) $f \in C^1(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$

b) $x\hat{f}(x) \in L^2(\mathbb{R})$.

Proof of a) implies b): we claim that $ix\hat{f}(x) = (f')\hat{}$ from which b) follows. We have $f^2(x) - f^2(0) = 2\int\limits_{0}^{x} f(t)f'(t)dt$. Since f and f' belong to $L^2(\mathbb{R})$ it follows that ff' is integrable so the right side has a finite limit l as $x \to \infty$. It follows that $f^2(x) \to l + f^2(0)$ as $x \to \infty$. The fact that $f^2 \in L^1(\mathbb{R})$ show that $l + f^2(0) = 0$ and $f(x) \to 0$ as $x \to \infty$. Similarly $f(x) \to 0$ as $x \to -\infty$. Now consider $\int\limits_{-\Delta}^{\Delta} f'(t)e^{-itx}dt = f(t)e^{-itx}|_{-\Delta}^{\Delta} + it\int\limits_{-\Delta}^{\Delta} f(t)e^{-itx}dt$. Letting $\Delta \to \infty$ we get $(f')\hat{}(t) = 0 + it\hat{f}(t)$ since $\int\limits_{-\Delta}^{\Delta} f'(t)e^{-itx}dt \to (f')\hat{}(t)$ in $L^2(\mathbb{R})$ and $\int\limits_{-\Delta}^{\Delta} f(t)e^{-itx}dt \to \hat{f}(t)$ in $L^2 \int\limits_{-\Delta}^{\Delta} f(t)e^{-itx}dt$.

b) implies a): $\hat{f}$ and $x\hat{f}(x)$ belong to $L^2(\mathbb{R})$ and hence $\hat{f} \in L^1(\mathbb{R})$. [Indeed $\int \left|\hat{f}\right| \leq \int\limits_{\{|x|\leq 1\}} \left|\hat{f}\right| + \int\limits_{\{|x|>1\}} \frac{1}{|x|}\{\left|x\hat{f}\right|\}$ and both terms are finite by Holder"s inequality]. Hence $f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \hat{f}(t)e^{itx}dx \ \forall x$ and $f'(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} it\hat{f}(t)e^{itx}dx$ by a simple application of Dominated Convergence Theorem. Thus f is continuously differentiable and $f'(-x)$ is $\hat{g}(x)$ where $g(x) = ix\hat{f}(x)$. Since $g \in L^2(\mathbb{R})$ we get $f' \in L^2(\mathbb{R})$.

Problem 540

Prove the following version of the Inversion Formula For Fourier Transforms of L^1 Functions:

Let $f \in L^1(\mathbb{R})$. Fix $x \in \mathbb{R}$. Show that $\lim\limits_{\Delta\to\infty} \int\limits_{-\Delta}^{\Delta} e^{itx}\hat{f}(t)dt = 0$ if $\frac{f(x+t)+f(x-t)}{t}$ is integrable on $(-\delta, \delta)$ for some $\delta > 0$. Hence show that if $\frac{f(x+t)+f(x-t)-2f(x)}{t}$ is integrable on $(-\delta, \delta)$ for some $\delta > 0$ then $\lim\limits_{\Delta\to\infty} \int\limits_{-\Delta}^{\Delta} e^{itx}\hat{f}(t)dt = f(x)$.

Remark: if f is differentiable at x then the integrability condition is satisfied. What we have here is the Dini"s test for convergence of Fourier integrals. Jordan"s test is also available; c.f. Pinsky book or Titchmarsh (Introduction to Theory of Fourier Integrals).

Note that if $f \in L^1(\mathbb{R})$ and $\hat{f} \equiv 0$ then $f * \phi_n$ is differentiable at 0 and its Fourier transform vanishes identically, where $\phi_n(x) = \sqrt{\frac{n}{2\pi}}e^{-nx^2/2}$. It follows by this problem that $f * \phi_n = 0 \ \forall n$ and since $f * \phi_n \to f$ in $L^1(\mathbb{R})$ we see that $f = 0$. Thus a uniqueness theorem for Fourier transforms of $L^1(\mathbb{R})$ functions follows from this problem.

We have $\int\limits_{-\Delta}^{\Delta} e^{itx}\hat{f}(t)dt = \int\limits_{-\Delta}^{\Delta} e^{itx} \int\limits_{-\infty}^{\infty} f(y)e^{-ity}d\mu(y)dt = \int\limits_{-\infty}^{\infty} f(y) \int\limits_{-\Delta}^{\Delta} e^{-ity}e^{itx}dtd\mu(y) =$
$\int\limits_{-\infty}^{\infty} f(y)\frac{2\sin\Delta(x-y)}{(x-y)}d\mu(y)$

$= \int\limits_{-\infty}^{\infty} f(x-y)\frac{2\sin\Delta y}{y}d\mu(y)$. Clearly this implies $\int\limits_{-\Delta}^{\Delta} e^{itx}\hat{f}(t)dt = \int\limits_{-\infty}^{\infty} \frac{f(x-y)+f(x+y)}{2}\frac{2\sin\Delta y}{y}d\mu(y) \to$ 0 by Riemann Lebesgue Lemma since $\frac{f(x-y)+f(x+y)}{2y}$ is integrable on $(-\delta, \delta)$ as well as on the complement of this interval.

Now suppose $\frac{f(x+t)+f(x-t)-2f(x)}{t}$ is integrable on $(-\delta,\delta)$ for some $\delta>0$. Let $g(t)=f(t)-f(x)e^{x^2/2}e^{-t^2/2}$. We claim that the first part applies to g in place of f so $\lim\limits_{\Delta\to\infty}\int\limits_{-\Delta}^{\Delta} e^{itx}\hat{g}(t)dt=0$; Since $\hat{g}(t)=\hat{f}(t)-f(x)e^{x^2/2}e^{-t^2/2}$ we get $\lim\limits_{\Delta\to\infty}\int\limits_{-\Delta}^{\Delta} e^{itx}\hat{f}(t)dt=f(x)e^{x^2/2}\lim\limits_{\Delta\to\infty}\int\limits_{-\Delta}^{\Delta} e^{itx}e^{-t^2/2}dt=f(x)$. It remains to verify that $\frac{g(x+t)+g(x-t)}{t}$ is integrable on $(-\delta,\delta)$ for some $\delta>0$. Since $\frac{g(x+t)+g(x-t)}{t}=\frac{f(x+t)+f(x-t)}{t}-f(x)e^{x^2/2}\frac{e^{-(x+t)^2/2}+e^{-(x-t)^2/2}}{t}=\frac{f(x+t)+f(x-t)}{t}-f(x)\frac{e^{-t^2/2}[e^{xt}+e^{-xt}]}{t}$ we only have to show that $\frac{2}{t}-\frac{e^{-t^2/2}[e^{xt}+e^{-xt}]}{t}$ is integrable in a neighbourhood of 0. This is clearly true.

Problem 541

Let $f\in L^2(\mathbb{R})$ and $S_\Delta f(x)=\int\limits_{-\Delta}^{\Delta}\hat{f}(x)e^{itx}d\mu(x)$. Show that $\int\limits_a^b S_\Delta f(x)dx\to\int\limits_a^b f(x)dx$ whenever $-\infty<a<b<\infty$.

A simple Fubini argument shows $<S_\Delta f,g>=<f,S_\Delta g>\ \forall f,g\in L^1(\mathbb{R})$. Note that $\|S_\Delta f\|_2=\left\|I_{[-\Delta,\Delta]}\hat{f}\right\|_2\le\left\|\hat{f}\right\|_2=\|f\|_2$. It follows that the formula $<S_\Delta f,g>=<f,S_\Delta g>$ holds $\forall f,g\in L^2(\mathbb{R})$. Now put $g=I_{(a,b)}$. We get $\int\limits_a^b S_\Delta f(x)dx=<S_\Delta f,g>=<f,S_\Delta g>$. Now $S_\Delta g\to g$ in $L^2(\mathbb{R})$ because $I_{[-\Delta,\Delta]}\hat{g}\to\hat{g}$ in $L^2(\mathbb{R})$ so $S_\Delta g(x)\to(\hat{g})\hat{}\,(-x)=g(x)$. Hence $<f,S_\Delta g>\to<f,g>=\int\limits_a^b f(x)dx$.

Problem 542

Let μ be a complex Borel measure on the unit circle T in the complex plane such that $\int z^{-n}d\mu(z)\to 0$ as $n\to\infty$. Show that $\int z^n d\mu(z)\to 0$ as $n\to\infty$.

In particular, if $f\in L^1(T)$ then $\hat{f}(n)\to 0$ as $n\to\infty$ iff $\hat{f}(n)\to 0$ as $n\to-\infty$.

Consider $\int z^{-n}p(z)d\mu(z)$ where $p(z)=\sum\limits_{k=-N}^{N}c_k z^k$ for some positive integer N and some complex numbers $c_{-N},\dots,c_N$. Since $\int z^{-n}z^k d\mu(z)\to 0$ as $n\to\infty$

for each k it follows that $\int z^{-n}p(z)d\mu(z) \to 0$. By Stone - Wierstrass Theorem polynomials form a dense subset of $C(T)$ w.r.t. the sup. norm. Hence $\int z^{-n}g(z)d\mu(z) \to 0$ for every $g \in C(T)$. Also any function in $L^1(|\mu|)$ can be approximated in the norm of this space by continuous functions. Hence $\int z^{-n}g(z)d\mu(z) \to 0$ for all $g \in L^\infty(|\mu|)$ (because $L^\infty(|\mu|) \subseteq L^1(|\mu|)$). It follows that $\int z^{-n}d\,|\mu|\,(z) \to 0$ which implies $\int z^n d\,|\mu|\,(z) \to 0$ as $n \to \infty$ [since $\bar{z}^{-n} = z^n$]. Repeating the argument above we get $\int z^n g(z) d\,|\mu|\,(z) \to 0$ for every $g \in L^\infty(|\mu|)$. Hence $\int z^n d\mu(z) \to 0$ as $n \to \infty$.

Problem 543

Let f and g be complex valued continuous function from $[0,\infty)$. If $\int\limits_0^x f(x-y)g(y) = 0\ \forall y \geq 0$ show that either $f \equiv 0$ or $g \equiv 0$. Give an example of continuous integrable functions f and g on $\mathbb{R}$ such that $f * g = 0$ but neither f nor g vanishes identically.

We first give the counter-example: let $f(x) = \begin{cases} \frac{\sin^2 x}{x^2} \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}$ and $g(x) = e^{iax}f(x)$. Then the Fourier transform of f is $\frac{1}{2\sqrt{2\pi}}(1 - \left|\frac{x}{2}\right|)^+$ as see easily using the inversion formula. In particular the support of $\hat{f}$ is contained in $[-2,2]$. Also $\hat{g}(t) = \hat{f}(t-a)$ which has support in $[a-2, a+2]$. If $a > 4$ we see that $\hat{f}\hat{g} \equiv 0$ which implies $f * g = 0$.

Now we come to the first part of the problem. We shall write $(f * g)(x) = \int\limits_0^x f(x-y)g(y)$. This coincides with the usual definition of convolution if f and g are integrable on $\mathbb{R}$ and both vanish on $(-\infty, 0)$. Let $f_1(x) = xf(x)$ and $g_1(x) = xg(x)$. We have $\int\limits_0^x (x-y)f(x-y)g(y)dy + \int\limits_0^x yg(y)f(x-y)dy = x\int\limits_0^x f(x-y)g(y)dy = 0$ or $\int\limits_0^x f_1(x-y)g(y)dy + \int\limits_0^x g_1(y)f(x-y)dy = 0$. Writing this as $f_1 * g + g_1 * f = 0$ we get $[f * g_1] * [f_1 * g + g_1 * f] = 0$ which says $(f_1 * g_1) * (f * g) + h * h = 0$ where $h = f * g_1$. Using the hypothesis again we get $h * h = 0$. **By the next problem** this implies $h = 0$ and hence $f * g_1 = 0$. Repeating this argument we get $f * g_n = 0$ for all n where $g_n(x) = x^n g(x)$. Thus $\int\limits_0^x y^n g(y)f(x-y)dy = 0$ for all n. By Wierstrass Theorem it follows that $y^n g(y) f(x-y) = 0$ for $0 \leq y \leq x, x \geq 0$. This implies either $f \equiv 0$ or $g \equiv 0$. [Given $t, s > 0$ put $x = t + s$ and $y = t$ to get $g(t)f(s) = 0\ \forall t, s > 0$].

Problem 544

If f is continuous on $[0, 2\Delta]$ and $\int\limits_0^x f(x-y)f(y)dy = 0$ for $0 \leq x \leq 2\Delta$ show that f vanishes identically on $[0, \Delta]$.

Remark: if we define f to be 0 on $\mathbb{R}\backslash[0, 2\Delta]$ it does not follow that $f * f = 0$ so theory of Fourier transforms does not yield this result immediately.

[Proof from Yosida's book]. The proof requires some preliminaries.

Lemma

$\lim\limits_{j\to\infty} \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int\limits_0^\Delta e^{kj(x-y)} f(y)dy = \int\limits_0^x f(y)dy$ if f is continuous on $[0, \Delta]$ and $0 \leq x \leq \Delta$.

By an easy application of Fubini's Theorem $\sum\limits_{k=0}^{\infty} \frac{(-1)^{k-1}}{k!} \int\limits_0^\Delta e^{kj(x-y)} f(y)dy = \int\limits_0^\Delta \sum\limits_{k=0}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kj(x-y)} f(y)dy$

$= -\int\limits_0^\Delta e^{-e^{j(x-y)}} f(y)dy \to -\int\limits_x^\Delta f(y)dy$ as $j \to \infty$. Hence $\lim\limits_{j\to\infty} \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int\limits_0^\Delta e^{kj(x-y)} f(y)dy = -\int\limits_x^\Delta f(y)dy + \int\limits_0^\Delta f(y)dy = \int\limits_0^x f(y)dy$

Lemma

If f is continuous on $[0, \Delta]$ and $\sup\limits_n \left| \int\limits_0^\Delta e^{ny} f(y)dy \right| < \infty$ then $f \equiv 0$.

In previous lemma we can replace $f(y)$ by $f(\Delta - y)$. We get $\lim\limits_{j\to\infty} \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int\limits_0^\Delta e^{kj(x-y)} f(\Delta - y)dy = \int\limits_0^x f(\Delta - y)dy$ or $\lim\limits_{j\to\infty} \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{jk(x-\Delta)} \int\limits_0^\Delta e^{kj(\Delta-y)} f(\Delta - y)dy = \int\limits_0^t f(\Delta - y)dy$. But $\left| \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{jk(x-\Delta)} \int\limits_0^\Delta e^{kj(\Delta-y)} f(\Delta - y)dy \right| \leq M \left| \sum\limits_{k=1}^{\infty} \frac{1}{k!} e^{jk(x-\Delta)} \right|$

$= M(e^{e^{j(x-\Delta)}} - 1) \to 0$ as $j \to \infty$ if $0 \leq x < \Delta$. Hence $\int\limits_0^x f(\Delta - y)dy = 0$

if $0 < x < \Delta$. In other words, $\int\limits_{\Delta - x}^{\Delta} f(z)dz = 0$if $0 < x < \Delta$ which forces f to vanish identically.

Now $\int\limits_0^{2\Delta} e^{j(2\Delta - x)} \int\limits_0^{x} f(x-y)f(y)dydx = 0$. Put $u = \Delta - y$ and $v = 2\Delta - u - x = \Delta + y - x$. Note that $0 \leq y \leq x \leq 2\Delta$ is equivalent to $u + v \geq 0, u \leq \Delta$ and $v \leq \Delta$. Also, $y = \Delta - u$ and $x = 2\Delta - u - v$. Since the Jacobian of $(x, y) \to (u, v)$ is 1 we get $\int \int\limits_{\substack{u+v \geq 0 \\ u\Delta, v \leq \Delta}} e^{j(u+v)} f(\Delta - v)f(\Delta - u)dudv = 0$. Let $R = \{(u, v) : u + v \geq 0, u \leq \Delta, v \leq \Delta\}$ and $R_1 = \{(u, v) : u + v \leq 0, u \geq -\Delta, v \geq -\Delta\}$. Then $R \cup R_1 = \{(u, v) : -\Delta \leq u, v \leq \Delta\}$. Also $\int \int\limits_{R \cup R_1} e^{j(u+v)} f(\Delta - v)f(\Delta - u)dudv = (\int\limits_{-\Delta}^{\Delta} e^{ju} f(\Delta - u)du)^2$. Hence $(\int\limits_{-\Delta}^{\Delta} e^{ju} f(\Delta - u)du)^2 = \iint\limits_R e^{j(u+v)} f(\Delta - v)f(\Delta - u)dudv + \iint\limits_{R_1} e^{j(u+v)} f(\Delta - v)f(\Delta - u)dudv$

$= \iint\limits_{R_1} e^{j(u+v)} f(\Delta - v)f(\Delta - u)dudv \leq \iint\limits_{R_1} f(\Delta - v)f(\Delta - u)dudv = (\int\limits_{-\Delta}^{\Delta} f(\Delta - y)dy)^2 \leq C\Delta^2$ where $C/2$ is an upper bound for $|f|$ on $[0, 2\Delta]$. We now have $\left|\int\limits_{-\Delta}^{\Delta} e^{ju} f(\Delta - u)du\right| \leq \sqrt{C}\Delta$. Since $\left|\int\limits_{-\Delta}^{0} e^{ju} f(\Delta - u)du\right| \leq \left|\int\limits_{-\Delta}^{0} f(\Delta - u)du\right| = \left|\int\limits_0^{2\Delta} f(t)dt\right| \leq C\Delta$ we have $\left|\int\limits_0^{\Delta} e^{ju} f(\Delta - u)du\right| \leq (C + \sqrt{C})\Delta$. By the second lemma above we conclude that $f(\Delta - u) = 0 \; \forall u \in [0, \Delta]$.

Problem 545

If $f : T \to T$ satisfies the equation $f(z^2) = f^2(z) \; \forall z \in T$ show that there is an integer n such that $f(z) = z^n \; \forall z \in T$.

Remark: this is stronger then the statement that the only continuous characters of T are the functions $z \to z^n$.

There is a continuos function $g : \mathbb{R} \to \mathbb{R}$ such that $f(e^{i2\pi t}) = e^{2\pi i g(t)}$ and $g(0) = 0$. [Theorem 7.6.2, p. 241 of "A Course in Probability Theory" by

Chung]. Claim: there is an integer n such that $g(x) = nx\ \forall x$. For this we observe two facts:

1) $g(t+1) - g(t)$ is an integer n independent of t

2) $g(2t) - 2g(t)$ is an integer m independent of t

These facts are easy: $e^{2\pi i[g(t+1)-g(t)]} = \frac{f(e^{2\pi i(t+1)})}{f(e^{2\pi it})} = 1$ and $e^{2\pi i[g(2t)-2g(t)]} = \frac{f(e^{i4\pi t})}{[f(e^{i2\pi t})]^2} = 1$ by hypothesis. The fact that the integer values in 1) and 2) do not depend on t follows by continuity. Let $h(t) = g(t) - nt + m$. Then $h(t+1) = h(t)$ and $h(2t) = 2h(t)$. This gives $h(\frac{t}{2^k}) = \frac{1}{2^k}h(t)$ so $h(s + \frac{1}{2^k}) = \frac{1}{2^k}h(2^k s+1) = \frac{1}{2^k}h(2^k s) = h(s)$. Iteration gives $h(s+\frac{j}{2^k}) = h(s)$ and continuity implies $h(s+x) = h(s)\ \forall s, x \in \mathbb{R}$. It follows that h is a constant which has to be 0. Thus $g(t) = nt + m$ and $f(e^{2\pi it}) = e^{2\pi i g(t)} = e^{2\pi int}$ or $f(z) = z^n$.

Problem 546

If μ is a positive Borel measure on $\mathbb{R}$ such that $\mu_t << \mu$ for every real number t, where $\mu_t(E) = \mu(E+t)$, show that μ is absolutely continuous w.r.t. Lebesgue measure.

We have $\int \mu(E-x)dx = \int\int I_{\{(x,y):x+y\in E\}} d(\mu\times m)(x,y) = \int m(E)d\mu(y) = 0$ if $m(E) = 0$. Hence there exists x such that $\mu(E-x) = 0$. By hypothesis this implies $\mu(E) = 0$.

Problem 547

Prove that the positive finite measures $\mu_n = \frac{1}{n}\sum\limits_{k=1, k \text{ even}}^{n} (-1)^k \delta_{k/n}$ converge to $\frac{1}{2}\lambda$ weakly (i.e. in the weak* topology of $C^*[0,1]$) and that the positive finite measures $\nu_n = \frac{1}{n}\sum\limits_{k=1, k \text{ odd}}^{n} (-1)^{k+1}\delta_{k/n}$ also converge to $\frac{1}{2}\lambda$ weakly where λ is Lebesgue measure on $(0,1)$. Conclude that $\frac{1}{n}\sum\limits_{k=1}^{n}(-1)^k f(\frac{k}{n}) \to 0$ as $n \to \infty$ for every $f \in C[0,1]$.

Note that $\mu_n([0,1] \to \frac{1}{2}$ and $\nu_n([0,1] \to \frac{1}{2}$. Hence, using standard arguments in Probability Theory it suffices to show that $\frac{1}{n}\sum\limits_{k=1, k \text{ even}}^{n} (-1)^k e^{itk/n} \to \frac{1}{2}\frac{e^{it}-1}{it}$ and $\frac{1}{n}\sum\limits_{k=1, k \text{ odd}}^{n} (-1)^k e^{itk/n} \to \frac{1}{2}\frac{e^{it}-1}{it}\ \forall t$. These facts can be proved by direct computation of the geometric sums involved.

Remark: the second part can be proved more easily as follows. Any $f \in C[0,1]$ extends to a continuous . function F on $[0, 2\pi]$ with $F(0) = F(2\pi)$. It follows from Fejer's theorem that f can be approximated uniformly by trigonometric polynomials. Since $\left|\frac{1}{n}\sum_{k=1}^{n}(-1)^k f(\frac{k}{n}) - \frac{1}{n}\sum_{k=1}^{n}(-1)^k g(\frac{k}{n})\right| \leq \|f-g\|_\infty$ it suffices to prove the result when $f(x) = e^{imx}$ for some integer m. In this case the convergence of $\frac{1}{n}\sum_{k=1}^{n}(-1)^k f(\frac{k}{n})$ to 0 is seen easily by explicit computation of $\frac{1}{n}\sum_{k=1}^{n}(-1)^k f(\frac{k}{n})$.

Remark: by the method of above remark we can also show that $\frac{1}{2^n}\sum_{k=1}^{n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix} f(\frac{k}{n}) \to 0$ as $n \to \infty$ for every $f \in C[0,1]$.

Problem 548

Let $f, f', f_n, f'_n(n = 1, 2, ...)$ and g be continuous on $[0,1]$. If $f_n \to f$ pointwise and $f'_n \to g$ pointwise show that $f' = g$ on $[0,1]$.

Suppose $f'(\alpha) \neq g(\alpha)$. Let $\delta > 0$ and $[a,b]$ be an interval containing α such that $a < b$ and $|f' - g| > \delta$ on $[a,b]$. We claim that $\{f'_n\}$ is uniformly bounded in some interval $[c,d] \subseteq [a,b]$ with $c < d$. To see this we write $[a,b]$ as $\bigcup_{k=1}^{\infty}\bigcap_{m=k}^{\infty}\{x \in [a,b] : |f'_m(x) - g(x)| \leq 1\}$. By Baire Category Theorem there exists an interval $[c,d] \subseteq [a,b]$ with $c < d$ and $[c,d] \subseteq \bigcap_{m=k}^{\infty}\{x \in [a,b] : |f'_m(x) - g(x)| \leq 1\}$. It follows that $|f'_m(x)| \leq 1 + \|g\|_\infty$ $\forall x \in [c,d]$ $\forall m \geq k$. This proves our claim. Let $[c_1, d_1] \subseteq [c,d]$. Let By DCT we now get $\int_{c_1}^{d_1} f'_n(y)dy \to \int_{c_1}^{d_1} g(y)dy$ or $f(d_1) - f(c_1) = \lim[f_n(d_1) - f_n(c_1)] = \lim \int_{c_1}^{d_1} f'_n(y)dy = \int_{c_1}^{d_1} g(y)dy$. This implies $f' = g$ on $[c,d]$, a contradiction.

Problem 549 [de Bois -Reymond Lemma]

Let $f, g \in C[0,1]$ and $\int_0^1 [fh' + gh] = 0$ for any continuously differentiable function h such that $h(0) = h(1) = 0$. Show that $f'(x)$ exists and equals $g(x)$ $\forall x \in [0,1]$.

Proof: if the functions involved are complex valued we can reduce the proof to the real case. So assume that f and g are real valued

Step 1: Let $f \in C[0,1]$ and $\int_0^1 fh = 0$ for any continuous h such that $h(0) = h(1) = 0$. Then $f \equiv 0$.

If $f > 0$ on $(\alpha, \beta) \subseteq [0,1]$ with $\alpha < \beta$ take $h(x) = (x-\alpha)(\beta - x)$ for $\alpha < x < \beta$ and 0 elsewhere to get a contradiction.

Step 2: Let $f \in C[0,1]$ and $\int_0^1 fh' = 0$ for any continuously differentiable function h such that $h(0) = h(1) = 0$. Then f is necessarily a constant.

Let $c = \int_0^1 f$ and $h(x) = \int_0^x \{f(t) - c\}dt$. Then h satisfies the conditions in the hypothesis, so $\int_0^1 fh' = 0$. This also implies $\int_0^1 (f-c)h' = 0$ and hence $\int_0^1 (f-c)^2 = 0$. Hence $f(x) = c \; \forall x$.

Step 3

Let $f, g \in C[0,1]$ and $\int_0^1 [fh' + gh] = 0$ for any continuously differentiable function h such that $h(0) = h(1) = 0$.

Let $\phi(x) = f(x) - f(0) - \int_0^x g$. Then ϕ is continuous and $\int_0^1 \phi h' = \int_0^1 fh' - \int_0^1 \int_0^x g(t)dth'(x)dx$. Now $\int_0^1 \int_0^x g(t)'h'(x)dx = \int_0^x g(t)dth(x)|_0^1 - \int_0^1 g(x)h(x)dx$. Hence $\int_0^1 \phi h' = \int_0^1 fh' + \int_0^1 gh = 0$ for any continuously differentiable function h such that $h(0) = h(1) = 0$, by hypothesis. By Step 2 ϕ is a constant,say C. Hence $f(x) - f(0) - \int_0^x g = C$ which implies that f is differentiable and $f' = g$.

Problem 550

Let x be a real number such that n^x is an integer for each $n \in \mathbb{N}$. Prove that $x \in \{0, 1, 2, ...\}$.

If $0 < x < 1$ then $(n+1)^x - n^x = x\xi^{x-1}$ for some $\xi \in (n, n+1)$. But $0 < x\xi^{x-1} \leq xn^{x-1} < 1$ if n is sufficiently large. This is a contradiction to the fact that $(n+1)^x - n^x$ is an integer. Thus, $x = 0$ or $x \geq 1$. Suppose $1 < x < 2$. Then $(n+2)^x - 2(n+1)^x + n^x = x(x-1)x\xi^{x-2}$ for some $\xi \in (n, n+2)$ and we can argue as above to get a contradiction. Clearly, the following version of MVT is all that is required to complete the proof: let $f : (0, \infty) \to \mathbb{R}$ be a C^∞ function define $Tf(x) = f(x+1) - f(x)$. Then $T^n f(x) = f^{(n)}(\xi)$ for some $\xi \in (x, x+n)$. The proof of this is by repeated application of the standard MVT.

Problem 551

Give an example of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that both of these functions have intermediate value property (IVP) but their sum $f + g$ does not.

We use the facts that any derivative has IVP and the square of a function with IVP has IVP. [See solution to Problem 416 above for the first fact. The second fact is obvious]. Let $f(x) = [\frac{d}{dx}\{x^2 \sin \frac{1}{x}\}]^2$ and $f(x) = [\frac{d}{dx}\{x^2 \cos \frac{1}{x}\}]^2$. Then f and g have IVP and $f(x) + g(x) = [2x \sin \frac{1}{x} - \cos \frac{1}{x}]^2 + [2x \cos \frac{1}{x} + \sin \frac{1}{x}]^2 = 4x^2 + 1$ [with $f(0) + g(0) = 0 + 0 = 0$]. Obviously, $f + g$ does not have IVP.

Problem 552

Give an elementary argument to show that the only locally integrable additive functions from $\mathbb{R}$ into itself are multiples of the identity map.

Remark: the only measurable additive functions from $\mathbb{R}$ into itself are multiples of the identity map. [see Problem 79 above]. We are asking for a simple proof which does not use the fact that $E - E$ contains an interval around 0 if E has positive measure.

Integrate the equation $f(x+y) = f(x) + f(y)$, where $x, y > 0$ w.r.t. x from 0 to t to get $\int_y^{t+y} f(x)dx = \int_0^t f(x)dx + tf(y)$. We claim that $\int_y^{t+y} f(x)dx - \int_0^t f(x)dx$ is symmetric in t and t, i.e. $\int_y^{t+y} f(x)dx - \int_0^t f(x)dx = \int_t^{t+y} f(x)dx - \int_0^y f(x)dx$. This is easily verified by considering the cases $y < t$ and $t \leq y$ separately. It follows that $tf(y) = yf(t)\ \forall t, y > 0$. Hence $tf(1) = f(t)\ \forall t > 0$ which completes the proof since $f(-x) = -f(x)$.

Problem 553

If $f : \mathbb{R} \to \mathbb{R}$ is additive and not continuous show that its graph is dense in $\mathbb{R}^2$.

Pick a such that $f(a) \neq af(1)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map whose matrix is $\begin{pmatrix} 1 & a \\ f(1) & f(a) \end{pmatrix}$. [In other words, $T(x,y) = (x+ay, f(1)x+f(a)y)$]. Then T is non-singular and hence a homeomorphism. Hence $\{T(x,y) : x,y \in \mathbb{Q}\}$ is dense. If $x,y \in \mathbb{Q}$ then $T(x,y) = (x+ay, f(1)x+f(a)y)$ and $f(1)x+f(a)y = f(x+ay)$ because x and y are rational. Thus, the graph of f contains the dense set$\{T(x,y) : x,y \in \mathbb{Q}\}$.

Problem 554

Let f and g be continuous functions on $[0,1]$ such that $\int(fh' + gh) = 0$ for every continuously differentiable function h such that $h(0) = h(1) = 0$. Show that f is differentiable and $f' = g$.

Remark: this is a basic lemma in Calculus of Variations.

Suppose $f(x) - f(0) - \int\limits_0^x g(t)dt > 0$ on some interval (α, β) with $\alpha < \beta$. We construct a continuously differentiable function h such that $h(0) = h(1) = 0$, $h' > 0$ on (α, β) and $h = 0$ on $[0,1]\backslash(\alpha, \beta)$. Once this is done we get $\int\limits_\alpha^\beta [f(x) - f(0) - \int\limits_0^x g(t)dt]h' > 0$ which gives (by an integration by parts). $-\int gh - \int\limits_\alpha^\beta \int\limits_0^x f(t)dth' = -\int gh + \int\limits_\alpha^\beta gh > 0$ a contradiction. We can then conclude that $f(x) - f(0) - \int\limits_0^x g(t)dt \leq 0$ and , since f and g can be replaced by $-f$ and $-g$, we get $f(x) - f(0) - \int\limits_0^x g(t)dt = 0 \; \forall x$, completing the proof. A function h with desired properties is given by

$$h(x) = \begin{cases} 0 \text{ if } x < \alpha \text{ or } x > \beta \\ \int\limits_\alpha^x (y-\alpha)(\beta-y)dy - (x-\alpha)^2\phi(x) \text{ if } \alpha \leq x \geq \beta \end{cases}$$

where ϕ is a continuously differentiable function with the following properties: $\phi(\beta) = \frac{\int\limits_\alpha^x (y-\alpha)(\beta-y)dy}{(\beta-\alpha)^2}$ and $\phi'(\beta) = -2\frac{\int\limits_\alpha^x (y-\alpha)(\beta-y)dy}{(\beta-\alpha)^3}$.

Problem 555

If $f \in L^2(\mathbb{R})$ show that the linear space spanned by translates of f is dense in $L^2(\mathbb{R})$ if and only if $m\{t : \hat{f}(t) = 0\} = 0$.

Remark: the corresponding result for $L^1(\mathbb{R})$ is also true, but Segal has shown that for $L^p(\mathbb{R})$ with $1 < p < 2$ the corresponding result is not true.

Suppose translates of f span a dense set in $L^2(\mathbb{R})$ and $m\{t : \hat{f}(t) = 0\} > 0$. There exists a compact set $K \subseteq \{t : \hat{f}(t) = 0\}$ such that $m(K) > 0$. Consider $< \hat{I}_{-K}, \sum_{j=1}^{N} a_j f_{x_j} >=< I_K, (\sum_{j=1}^{N} a_j e^{-itx_j})\hat{f} >= 0$ whenever $N \geq 1, a_j \in \mathbb{R}, x_j \in \mathbb{R}$ for $1 \leq j \leq N$. It follows by hypothesis that $\hat{I}_{-K} = 0$. Hence $I_{-k} = 0$ contradicting the fact that $m(K) > 0$. This proves the "only if" part. For the converse suppose g is orthogonal to all the translates of f. Then $\int [\hat{g}(x)]^- \hat{f}(x) e^{-ixt} dx = \int [\hat{g}(x)]^- \hat{f}_t(x) dx =< \hat{f}_t, \hat{g} >=< f_t, g >= 0$. It follows that the Fourier transform of the integrable function $[\hat{g}(x)]^- \hat{f}(x)$ vanishes. Hence $[\hat{g}(x)]^- \hat{f}(x) = 0$ a.e. If $m\{t : \hat{f}(t) = 0\} = 0$ then we get $\hat{g} = 0$, hence $g = 0$ as required.

Problem 556

Let $T : L^p(\mu) \to L^p(\mu)$ be a linear map which maps non0negative functions to non-negative functions. Show that T is a bounded operator.

Suppose $\sup\{\|Tf\| : f \geq 0, \|f\| \leq 1\} = \infty$. Then there exists a sequence of non-negative functions $\{f_n\}$ such that $\|f_n\| \leq 1 \ \forall n$ and $\|Tf_n\| > n^2$. Let $f = \sum_{n=1}^{\infty} \frac{f_n}{n^2}$. The series converges in the norm of $L^p(\mu)$ and defines a non-negative function f in $L^p(\mu)$. Since $f \geq \sum_{n=1}^{N} \frac{f_n}{n^2}$ for any positive integer N the hypothesis shows $Tf \geq \sum_{n=1}^{N} \frac{Tf_n}{n^2}$. Hence $\int (Tf)^p d\mu \geq \int (\sum_{n=1}^{N} \frac{Tf_n}{n^2})^p d\mu \geq \int \sum_{n=1}^{N} (\frac{Tf_n}{n^2})^p d\mu = \sum_{n=1}^{N} \frac{\|Tf_n\|^p}{n^{2p}} > \sum_{n=1}^{N} \frac{n^{2p}}{n^{2p}} = N$. Since N is arbitrary we have arrived at a contradiction. Hence $\sup\{\|Tf\| : f \geq 0, \|f\| \leq 1\} < \infty$. Since any $f \in L^p(\mu)$ can be written as $f^+ - f^-$ and $\|f^+\| \leq \|f\|, \|f^-\| \leq \|f\|$ we get $\sup\{\|Tf\| : \|f\| \leq 1\} < \infty$ as required.

Problem 557

Prove that the following converse of Birkhoff's Ergodic Theorem is *false*:

If $(\Omega, \mathcal{F}, P)$ is a probability space, $T : \Omega \to \Omega$ is measurable and $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(T^k(x))$ exists almost everywhere for every $f \in L^1(P)$ then T is measure preserving (i.e. $P \circ T^{-1} = P$).

Let $(\Omega, \mathcal{F}, P)$ be the interval $[0, 1)$ with Borel sigma field and Lebesgue measure. Let $T : [0, \frac{1}{4}) \to [\frac{1}{4}, 1)$ and $T : [\frac{1}{4}, 1) \to [0, \frac{1}{4})$ be measurable maps such that $T^2 = I$. For example, let $T(x) = \frac{1}{4} + 3x$ on $[0, \frac{1}{4})$ and $T(x) = \frac{x}{3} - \frac{1}{12}$ on $[\frac{1}{4}, 1)$. Then $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(T^k(x)) = \frac{f(x)+f(T(x))}{2}$ $\forall x$ and $P(T^{-1}[\frac{1}{4}, 1)) = \frac{1}{4} \neq \frac{3}{4} = P\ ([\frac{1}{4}, 1))$. Note that $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f \circ T^k$ exists in $L^1(P)$ also.

Problem 558

Let $L^1(\mu)$ denote the space of *real valued* integrable functions w.r.t. a finite positive measure μ.

a) Suppose $f_n \to f$ weakly in $L^1(\mu), g_n \to g$ weakly in $L^1(\mu)$ and $|f_n| \leq g_n$ a.e.. Show that $|f| \leq g$ a.e..

b) Prove or disprove: $f_n \to f$ weakly in $L^1(\mu), g_n \to g$ weakly in $L^1(\mu)$ and $|f_n| \leq |g_n|$ a.e.. Show that $|f| \leq |g|$ a.e..

For any measurable set A we have $\int_A |f|\, d\mu = \int_A f\phi d\mu$ where $\phi(x) = \left\{ \begin{array}{c} 1 \text{ if } f(x) > 0 \\ 0 \text{ if } f(x) = 0 \\ -1 \text{ if } f(x) < 0 \end{array} \right.$. Since $I_A\phi \in L^\infty(\mu)$ we get $\int_A |f|\, d\mu = \lim_{n\to\infty} \int_A f_n \phi d\mu$. Hence $\int_A |f|\, d\mu \leq \limsup_{n\to\infty} \int_A g_n d\mu = \int_A g d\mu$. Hence $|f| \leq g$ a.e..

The statement in b) is false. For this we take a sequence $\{g_n\} \subseteq L^1(\mu)$ such that $|g_n| = 1$ a.e. and $g_n \to 0$ weakly. [On such sequence is provided by $g_n = 2I_{\{X_n=0\}} - 1$ where $X_n(\omega)$ is the $n-th$ coefficient in the expansion of $\omega \in (0, 1)$ w.r.t. base 2; the basic measure space is $(0, 1)$ with Borel sigma algebra and Lebesgue measure P; note that if $A \in \sigma\{X_1, X_2, , , , X_k\}$ for some k then $\int_A g_n d\mu = 2P\{A \cap \{X_n = 0\}) - P(A) = 2P(A)P\{X_n = 0\} - P(A) = 0$ whenever $n > k$ (by independence); Now given any Borel set A and $\varepsilon > 0$ then there exists $k \in \mathbb{N}$ and $B \in \sigma\{X_1, X_2, , , , X_k\}$ such that $P(A\Delta B) < \varepsilon$. Since

$\left|\int\limits_A g_n d\mu - \int\limits_B g_n d\mu\right| < \varepsilon$ it follows that $\int\limits_A g_n d\mu \to 0$]. If $f_n \equiv 1$ then $f_n \to 1$ weakly, $g_n \to g \equiv 0$ weakly, $|f_n| \le |g_n|$ a.e. for each n but $|f| > g$ everywhere.

Problem 559 [Riemann Lebesgue Lemma]

Let μ be a probability measure on $(\Omega, \mathcal{F})$ and let $\{f_n\}_{n\in\mathbb{Z}}$ be an orthonormal set in $L^2(\mu)$ such that $\sup\limits_n \|f_n\|_\infty < \infty$. Show that $\int f f_n d\mu \to 0$ as $n \to \infty$ for every $f \in L^1(\mu)$.

Remark: taking μ to be the normalized Lebesgue measure on $[0, 2\pi]$ and $f_n(x) = e^{inx}, n \in \mathbb{Z}$ we get the usual Riemann Lebesgue Lemma.

Proof: if $f \in L^2(\mu)$ then $\sum \left|< \bar{f}, f_n >\right|^2 < \infty$ so $\int f f_n d\mu \to 0$. Given $f \in L^1(\mu)$ and $\varepsilon > 0$ choose a bounded measurable function g such that $\|f - g\|_1 < \varepsilon$. Then $\int g f_n d\mu \to 0$ and $\left|\int f f_n d\mu - \int g f_n d\mu\right| \le \sup\limits_n \|f_n\|_\infty \varepsilon \; \forall n$.

Problem 560

Let μ be a finite measure, $\{f_n\} \subseteq L^1(\mu), f_n \to f$ a.e. and assume that given $\varepsilon > 0$ there exists $\delta > 0$ and a positive integer n_0 such that $\left|\int\limits_E f_n d\mu\right| < \varepsilon$ whenever $\mu(E) < \delta$ and $n \ge n_0$. Show that f is integrable.

Choose Δ such that $\mu\{x : |f(x)| > \Delta\} < \delta$. Then $\mu\{x : |f_n(x)| > 2\Delta\} < \delta$ for n sufficiently large because $\mu\{x : |f_n(x) - f(x)| > \Delta\} \to 0$ as $n \to \infty$. Hence $\left|\int\limits_{\{x:|f_n(x)|>2\Delta\}} f_n d\mu\right| < \varepsilon$ for n sufficiently large. This shows that the sequence $\{f_n\}$ is uniformly integrable and this implies $f \in L^1(\mu)$ and $f_n \to f$ in $L^1(\mu)$.

Problem 561

Prove or disprove: $\lim\limits_{t\to\infty} \int\limits_{-\infty}^{\infty} |f_t(x) - f(x)|\, dx$ exists for every $f \in L^1(\mathbb{R})$ where $f_t(x) = f(x - t)$.

If f is continuous with compact support then f_t and f have disjoint supports for $|t|$ sufficiently large. Hence $\int\limits_{-\infty}^{\infty} |f_t(x) - f(x)|\, dx = \int\limits_{-\infty}^{\infty} |f_t(x)|\, dx +$

$\int_{-\infty}^{\infty} |f(x)|\, dx = 2\int_{-\infty}^{\infty} |f(x)|\, dx$. Since continuous functions with compact support form a dense subset of $L^1(\mathbb{R})$ it follows that $\lim\limits_{t\to\infty} \int_{-\infty}^{\infty} |f_t(x) - f(x)|\, dx$ and $\lim\limits_{t\to-\infty} \int_{-\infty}^{\infty} |f_t(x) - f(x)|\, dx$ both exist and the both the limits are $2\int_{-\infty}^{\infty} |f(x)|\, dx$.

Problem 562
Show that $\frac{\pi}{2}|\sin(2\pi nx)| \to 1$ in the weak* topology of $L^\infty[0,1] \equiv (L^1[0,1])^*$.

Proof: if $a < b$ then $\int_a^b |\sin(2\pi nx)|\, dx = \frac{1}{2\pi n}\int_{2\pi na}^{2\pi nb} |\sin y|\, dy = \frac{1}{2\pi n}\sum \int_{2\pi j}^{2\pi(j+1)} |\sin y|\, dy + o(1)$ where the sum extends over all j such that $2\pi na < 2\pi j$ and $2\pi(j+1) < 2\pi nb$ i.e., $na < j$ and $j < nb - 1$. Since $\int_{2\pi j}^{2\pi(j+1)} |\sin y|\, dy = \int_0^{2\pi} |\sin y|\, dy = 4$ we see that $\int_a^b |\sin(2\pi nx)|\, dx \to \frac{2}{\pi}\lim\limits_{n} \frac{nb-1-na}{n} = \frac{2}{\pi}(b-a)$. This implies that $\int |\sin(2\pi nx)|\, f(x)dx \to \frac{2}{\pi}\int f(x)dx\ \forall f \in L^1[0,1]$.

Problem 563 [From Zaanen's "Integration"]

Suppose $|f| \leq g \in L^1[0,1]$. Show that there exists a sequence $\{f_n\}$ in $L^1[0,1]$ such that $\int_E f_n \to \int_E f$ and $\int_E |f_n| \to \int_E g$

for every Borel set E in $[0,1]$. (All functions are real valued).

Let $h_1 = \frac{f+g}{2}$ and $h_2 = \frac{g-f}{2}$. Let $f_n(x) = h_1(x)\{|\sin(2\pi nx)| + \sin(2\pi nx)\} - h_2(x)\{|\sin(2\pi nx)| - \sin(2\pi nx)\}$. Observe that $\phi_n(x) \equiv h_1(x)\{|\sin(2\pi nx)| + \sin(2\pi nx)\}$ and $\psi_n(x) \equiv h_2(x)\{|\sin(2\pi nx)| - \sin(2\pi nx)\}$ have disjoint supports and that h_1, h_2, ϕ_1, ϕ_2 are all non-negative. By previous problem $\int_E h_j(x)\, |\sin(2\pi nx)|\, dx \to \frac{2}{\pi}\int_E h_j(x)dx, j = 1, 2$. Also $\int_E h_j(x)\{\sin(2\pi nx)\}dx \to 0, j = 1, 2$ by Riemann Lebesgue Lemma. Hence $\int_E f_n = \int_E [\phi_n - \psi_n] \to \frac{2}{\pi}\int_E h_1(x)dx - \frac{2}{\pi}\int_E h_2(x)dx =$

$\frac{2}{\pi}\int\limits_E f$. Also, $\int\limits_E |f_n| = \int\limits_E [\phi_n + \psi_n] \to \frac{2}{\pi}\int\limits_E h_1(x)dx + \frac{2}{\pi}\int\limits_E h_2(x)dx = \frac{2}{\pi}\int\limits_E g$ where the first equality follows from the fact that ϕ_n and ψ_n are non-negative and have disjoint support. Replacing f_n by $\frac{\pi}{2}f_n$ we get the desired result.

Problem 564

Let (X, d) be a compact metric space. A homeomorphism $T : X \to X$ is called an expansion if $x \neq y$ implies $d(T^n x, T^n y) > \delta$ for some positive integer n. Show that this concept does not change if d is replaced by an equivalent metric.

We prove the equivalence of the following properties of T:

a) T is an expansion

b) there exists a finite open cover $\{U_1, U_2, ..., U_k\}$ of X such that $x, y \in \bigcap\limits_{n=-\infty}^{\infty} T^{-n}A_n$ and $A_n \in \{U_1, U_2, ..., U_k\}$ for each integer n implies $x = y$.

c) there exists a finite open cover $\{U_1, U_2, ..., U_k\}$ of X such that $x, y \in \bigcap\limits_{n=-\infty}^{\infty} T^{-n}\bar{A}_n$ and $A_n \in \{U_1, U_2, ..., U_k\}$ for each integer n implies $x = y$.

Suppose a) holds and let $\delta > 0$ be as in the definition of an expansion. Let $\{U_1, U_2, ..., U_k\}$ be a $\frac{\delta}{2}$ open cover for X. Suppose $A_n \in \{U_1, U_2, ..., U_k\}$ for each integer n. If $x, y \in \bigcap\limits_{n=-\infty}^{\infty} T^{-n}\bar{A}_n$ then, for each n, $T^n x, T^n y \in \bar{A}_n = \bar{U}_j$ for some $j (= j(n))$; sine the diameter of $\bar{U}_j \leq \delta$ we get $d(T^n x, T^n y) \leq \delta$ for every n. This implies $x = y$ and show that a) implies c). Obviously, c) implies b). Now let b) hold. The open cover $\{U_1, U_2, ..., U_k\}$ of the compact metric space X has a Lebesgue number δ. [This means any set whose diameter is less than δ is contained in one of the sets $U_1, U_2, ..., U_k$]. Now suppose $x, y \in X$ and $d(T^n x, T^n y) \leq \frac{\delta}{2}$ for every n. Then, for any n, the points $T^n x$ and $T^n y$ belong to some U_{i_n}. Let $A_n = U_{i_n}$. Then $x, y \in \bigcap\limits_{n=-\infty}^{\infty} T^{-n}A_n$ and hence $x = y$ by b). In other words $x \neq y$ implies $d(T^n x, T^n y) > \frac{\delta}{2}$ for some n.

Problem 565 [Continuation of Problem 564]

Show that if T is an expansion so is T^2.

Remark: the proof works for any T^k.

By b) of previous problem there exists a finite open cover $\{U_1, U_2, ..., U_k\}$ of X such that $x, y \in \bigcap\limits_{n=-\infty}^{\infty} T^{-n}A_n$ and $A_n \in \{U_1, U_2, ..., U_k\}$ for each integer

n implies $x = y$. Consider the open cover $\{U_i \cap T^{-1}U_j : 1 \leq i, j \leq k\}$. If $x, y \in \bigcap_{n=-\infty}^{\infty} T^{-2n}A_n$ and $A_n \in \{U_i \cap T^{-1}U_j : 1 \leq i, j \leq k\}$ for each integer n then, for each n, $T^{2n}x$ and $T^{2n}y \in A_n$ for some $A_n \in \{U_i \cap T^{-1}U_j : 1 \leq i, j \leq k\}$. Hence either $T^{2n}x$ and $T^{2n}y \in A_n$ for some $A_n \in \{U_1, U_2, ..., U_k\}$ and $T^{2n+1}x, T^{2n+1}y \in T(A_n)$ and $T(A_n) \in \{U_1, U_2, ..., U_k\}$. It follows that for every positive integer m the points $T^m x$ and $T^m y$

belong to a set from $\{U_1, U_2, ..., U_k\}$. Hence $x = y$.

Problem 566

If T is an operator on a Hilbert space H such that $\sup_n \|T^n\| < \infty$ show that $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^n x$ exists for every $x \in H$.

First assume that $T = T^*$. Let $M = \{xH : Tx = x\}$ and $N = (I - T)H$. If $x \in M$ then $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^n x = x$. Suppose $x \in N$. Then $x = y - Ty$ for some y and $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^n x = \lim_{n\to\infty} \frac{1}{n}[y - T^n y] = 0$ since $\sup_n \|T^n\| < \infty$. It follows from this and the hypothesis that $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^n x = 0$ for all $x \in \bar{N}$. We claim that $H = M + \bar{N}$. If u is orthogonal to N then $< u, x - Tx >= 0 \ \forall x$ which says $u - T^*u = 0$. Since $T = T^*$ we get $Tu = u$ and $u \in M$. Hence $H = \bar{N} + N^{\perp} \subseteq \bar{N} + M$. This proves the result when T is self adjoint. Since any bounded operator T can be written as $T_1 + iT_2$ with T_1 and T_2 self adjoint the proof in the general case is complete.

Problem 567 (Lie Product Formula)

Let X be a Banach space and $T, S : X \to X$ be bounded linear maps. Show that

$e^{(T+S)} = \lim_{n\to\infty} \{e^{\frac{T}{n}} e^{\frac{S}{n}}\}^n$ in operator norm. [Here e^T is defined by $e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}$].

Remark: if T and S are also self adjoin then it can be shown that

$e^{i(T+S)} = \lim_{n\to\infty} \{e^{\frac{iT}{n}} e^{\frac{iS}{n}}\}^n$. [Ref. Reed and Simon, Functional Analysis, p. 295]

Let $U_n = e^{\frac{T}{n}} e^{\frac{S}{n}}$ and $V_n = e^{\frac{T+S}{n}}$. Then $\|U_n^n - V_n^n\| = \left\| \sum_{k=0}^{n-1} U_n^k (U_n - V_n) V_n^{n-k-1} \right\| \leq$
$\sum_{k=0}^{n-1} \|U_n\|^k \|V_n\|^{n-k-1} \|U_n - V_n\|$

$\leq \sum_{k=0}^{n-1} \alpha_n^{k+n-k+1} \|U_n - V_n\|$ where α_n is the maximum of $\|U_n\|$ and $\|V_n\|$. Thus, $\|U_n^n - V_n^n\| \leq n\alpha_n^{n-1} \|U_n - V_n\|$

$\leq n \|U_n - V_n\| e^{\|T\|+\|S\|}$ since $\alpha_n^{n-1} = \max\{\|U_n\|^{n-1}, \|V_n\|^{n-1}\}, \|U_n\|^{n-1} = \left\|e^{\frac{T}{n}} e^{\frac{S}{n}}\right\|^{n-1} \leq e^{\frac{(n-1)\|T\|}{n}} e^{\frac{(n-1)\|S\|}{n}} \leq e^{\|T\|+\|S\|}$ and $\|V_n\|^{n-1} = \left\|e^{\frac{T+S}{n}}\right\|^{n-1} \leq e^{\|T+S\|} \leq e^{\|T\|+\|S\|}$. Now $\left\|e^{\frac{T}{n}} e^{\frac{S}{n}} - e^{\frac{T+S}{n}}\right\| \leq \frac{C}{n^2}$ for some constant C. [This follows by expanding the exponentials in their power series and noting that $e^{\frac{T}{n}} e^{\frac{S}{n}} = 1 + \frac{T+S}{n} + O(\frac{1}{n^2}), e^{\frac{T+S}{n}} = 1 + \frac{T+S}{n} + O(\frac{1}{n^2})$]. This gives $\|U_n^n - V_n^n\| \leq \frac{C}{n} e^{\|T\|+\|S\|} \to 0$. Since $V_n^n = e^{(T+S)}$ we are done.

Problem 568

If M is a closed subspace of a Banach space X and $x \in X$ is the infimum in $\|x + M\| = \inf\{\|x + y\| : y \in M\}$ always attained?

Remark: if X is a Hilbert space then the infimum is attained when $y = -P_M x$ where P_M is the orthogonal projection with range M.

The answer is NO. Let $X = C[0,1]$ and $M = \{g \in X : \int_0^{1/2} g = \int_{1/2}^1 g\}$.

Let $f(x) \equiv x$. Claim:$\|f + M\| = \frac{1}{4}$. If $g \in M$ then $\int_0^{1/2} \{x + g(x)\}dx - \int_{1/2}^1 \{x + g(x)\}dx = \frac{1}{8} - \frac{3}{8} = -\frac{1}{4}$. Hence $\frac{1}{4} \leq \|f + g\| (\frac{1}{2} + \frac{1}{2})$. Taking infimum over $g \in M$ we get $\|f + M\| \geq \frac{1}{4}$. Let μ be the real measure which has density $I_{[0,\frac{1}{2})} - I_{[\frac{1}{2},1]}$ w.r.t. Lebesgue measure. Since the norm of μ in $(C[0,1])^*$ is 1 there exists $\{\phi_n\} \subseteq C[0,1]$ such that $\|\phi_n\| = 1 \ \forall n$ and $\int \phi_n d\mu \to 1$. Let $a_n = -\frac{1}{4 \int \phi_n d\mu}$ and $g_n(x) = a_n \phi_n(x) - x$. Then $\|f + g_n\| = |a_n| \to \frac{1}{4}$ as $n \to \infty$. Since $\int_0^{1/2} g_n - \int_{1/2}^1 g_n = a_n\{\int_0^{1/2} \phi_n - \int_{1/2}^1 \phi_n\} + \frac{1}{4} = a_n \int \phi_n d\mu + \frac{1}{4} = 0$ we see that $g_n \in M$ $\forall n$. Since $\|f + M\| \leq \|f + g_n\| \to \frac{1}{4}$the claim is proved. Suppose there exists $h \in M$ such that $\|f + h\| = \frac{1}{4}$. Then $\int_0^{1/2} \{x + h(x)\}dx - \int_{1/2}^1 \{x + h(x)\}dx = -\frac{1}{4}$ so

$\frac{1}{4} \leq \|f+h\| = \frac{1}{4}$ which implies that $x + h(x) = \frac{1}{4}$ on $[0, \frac{1}{2})$ and $x + h(x) = -\frac{1}{4}$ on $[\frac{1}{2}, 1)$. This is a contradiction, so $\|f + M\|$ is not attained.

Problem 569

In a normed linear space show that $\|x+y\| = \|x\| + \|y\| \Rightarrow \|ax+by\| = a\|x\| + b\|y\| \ \forall a, b \geq 0$.

Assume $b \leq a$. Then $\|ax+by\| = \|a(x+y)+(b-a)y\| \geq a\|x+y\| - (a-b)\|y\| = a(\|x\|+\|y\|) - (a-b)\|y\|$
$= a\|x\| + b\|y\|$ and $\|ax+by\| \leq a\|x\| + b\|y\|$.

Problem 570

Let $\sum_{n=1}^{\infty} x_n$ be an unconditionally convergent series in a Banach space X. Define $T : l^\infty \to X$ by $T(\{a_n\}) = \sum_{n=1}^{\infty} a_n x_n$. Show that T is a well-defined bounded linear operator.

The fact that $\sum_{n=1}^{\infty} a_n x_n$ converges for each $\{a_n\} \in l^\infty$ is standard. See, for example, p 458, Lemma 16.1 of Bases In Banach Spaces by Singer. Let $T_N(x^*, \{a_n\}) = \sum_{n=1}^{N} a_n x^*(x_n)$ for $x^* \in X^*, N \geq 1$. Since $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for each $x^* \in X^*$ each T_N is a bounded operator on $X^* \times l^\infty$ with $\sup\{|T_N(x^*, \{a_n\})| : N \geq 1\} < \infty$. Uniform Boundedness Principle shows $\left|\sum_{n=1}^{N} a_n x^*(x_n)\right| \leq C \max\{\|x^*\|, \|\{a_n\}\|\}$ (with C independent of N). Hence $\left\|\sum_{n=1}^{N} a_n x_n\right\| \leq C\|\{a_n\}\| \ \forall N$. This gives $\|T\{a_n\}\| \leq C\|\{a_n\}\|$.

Problem 571

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded operator such that $\|Tx\| \geq c\|x\| \ \forall x \in X$ for some constant $c > 0$. Suppose $T_n : X \to Y, n \geq 1$ be a bounded operators such that $\|T_n - T\| \to 0$. If each T_n maps X onto Y show that T also maps X onto Y.

Let $y \in Y$ and choose $x_n \in X$ such that $T_n x_n = y$. Note that $\|T_n x\| \geq \|Tx\| - \|(T_n - T)x\|$
$\geq c\|x\| - \frac{c}{2}\|x\| = \frac{c}{2}\|x\| \ \forall x$. Now $\|x_n - x_m\| \leq \frac{1}{c}\|Tx_n - Tx_m\| \leq \frac{1}{c}\|Tx_n - T_n x_n\| + \frac{1}{c}\|T_m x_m - Tx_m\|$ because $T_m x_m = T_n x_n (= y)$. Hence $\|x_n - x_m\| \leq \frac{1}{c}\|T_n - T\|\|x_n\| + \frac{1}{c}\|T_m - T\|\|x_m\|$. Since $\|T_n x\| \geq \frac{c}{2}\|x\| \ \forall x$ we get $\|y\| = \|T_n x_n\| \geq \frac{c}{2}\|x_n\|$

and $\|y\| \geq \frac{c}{2}\|x_m\|$. It follows that $\{x_n\}$ is Cauchy. Let $x_n \to x$. Now $\|y - Tx\| = \|T_n x_n - Tx\| \leq \|T_n - T\|\,\|x_n\| + \|Tx_n - Tx\| \to 0$ so $y \in T(X)$.

Problem 572

Let $f \in L^1(\mathbb{R})$ and suppose f vanishes outside $[-\Delta, \Delta]$. Let $g(x) = \int\limits_{-\infty}^{x} f(t)dt$. Show that $\frac{g(x+h)-g(x)}{h} \to f$ in $L^1(\mathbb{R})$ as $h \to 0$.

We consider positive values of h. A similar argument can be used when $h \to 0$ through negative values. If f is also continuous then $\int\limits_{-\infty}^{\infty} \left|\frac{g(x+h)-g(x)}{h} - f(x)\right| dx =$

$$\int\limits_{-\infty}^{\infty} \left|\frac{1}{h}\int\limits_{x}^{x+h} f(t)dt - f(x)\right| dx = \int\limits_{-\infty}^{\infty} \left|\frac{1}{h}\int\limits_{x}^{x+h} [f(t) - f(x)]dt\right| dx$$

$$\leq \frac{1}{h}\int\limits_{-\infty}^{\infty}\int\limits_{t-h}^{t} |f(t) - f(x)|\, dxdt = \frac{1}{h}\int\limits_{-\Delta-1}^{\Delta+1}\int\limits_{t-h}^{t} |f(t) - f(x)|\, dxdt \text{ if } 0 < h < 1.$$

Since f is uniformly continuous and $\frac{1}{h}\int\limits_{-\Delta-1}^{\Delta+1}\int\limits_{t-h}^{t} 1dxdt = \Delta{+}2 < \infty$ it follows that $\int\limits_{-\infty}^{\infty} \left|\frac{g(x+h)-g(x)}{h} - f(x)\right| dx \to 0$. For the general case choose $\psi \in C_c(\mathbb{R})$ such that $\int\limits_{-\infty}^{\infty} |f(x) - \psi(x)|\, dx < \varepsilon$. Let $\phi(x) = \int\limits_{-\infty}^{x} \psi(t)dt$. Then $\int\limits_{-\infty}^{\infty} \left|\frac{g(x+h)-g(x)}{h} - \frac{\phi(x+h)-\phi(x)}{h}\right| dx$

$\leq \int\limits_{-\infty}^{\infty} \left|\frac{1}{h}\int\limits_{x}^{x+h} \{f(t) - \psi(t)\}dt\right| dx = \frac{1}{h}\int\limits_{-\infty}^{\infty} |f(t) - \psi(t)| \int\limits_{t-h}^{t} dxdt < \varepsilon$ so $\int\limits_{-\infty}^{\infty} \left|\frac{g(x+h)-g(x)}{h} - f(x)\right| dx <$ $2\varepsilon + \int\limits_{-\infty}^{\infty} \left|\frac{\phi(x+h)-\phi(x)}{h} - \psi(x)\right| dx < 3\varepsilon$ if $h > 0$ is sufficiently small.

Problem 573

If $1 < p < \infty$, $f_n \to f$ a.e. and $\{\|f_n\|_p\}$ is bounded show that $f_n \to f$ weakly in L^p.

Remarks: the conclusion fails in L^1: $f_n = nI_{(0,\frac{1}{n})}, f = 0$. However, if the boundedness of $\{\|f_n\|_1\}$ is replaced by the condition $\|f_n\|_1 \to \|f\|_1$ then $\{f_n\}$ not only converges weakly, it converges in L^1.

There exists a finite constant C such that $\|f\|_p$ and $\|f\|_p$ do not exceed C for any n. Fix $g \in L^q$. By DCT $\int\limits_{\{|g| \leq \delta\}} |g|^q \to 0$ as $\delta \to 0$. Let $\varepsilon > 0$ and let $A = \{|g| > \delta\}$ where δ is chosen such that $\int\limits_{\{|g| \leq \delta\}} |g|^q < \frac{\varepsilon}{C}$. Note that $\mu(A) < \infty$. There exists $B \subseteq A$ such that $f_n \to f$ uniformly on B and $\mu(A \backslash B) < \eta$ where $0 < \eta < \delta$is such that $\int\limits_E |g|^q < \varepsilon$ whenever $\mu(E) < \eta$. . Let n_0 be such that $n \geq n_0$ and $x \in B$ implies $|f_n(x) - f(x)| < \frac{\varepsilon}{\|g\|_q \{\mu(B)\}^{1/p}}$. We now have $\left|\int f_n g - \int fg\right| \leq \int\limits_{A \backslash B} |f_n - f|\,|g| + \int\limits_{A^c} |f_n - f|\,|g| + \int\limits_B |f_n - f|\,|g|$. Note that $\int\limits_B |f_n - f|\,|g| \leq \frac{\varepsilon}{\|g\|_q \{\mu(B)\}^{1/p}} \int\limits_B |g| \leq \varepsilon$ by Holder's inequality. Next we note that $\int\limits_{A \backslash B} |f_n - f|\,|g| \leq 2C(\int\limits_{A \backslash B} |g|^q)^{1/q} < 2C\varepsilon$. Finally, $\int\limits_{A^c} |f_n - f|\,|g| \leq 2C(\int\limits_{A^c} |g|^q)^{1/q} < 2C(\frac{\varepsilon}{C})^{1/q}$. It follows that $\left|\int f_n g - \int fg\right| \leq \varepsilon + 2C\varepsilon + 2C^{1/p}\varepsilon^{1/q}$.

[To prove the result in the remark above first note that $\limsup \int\limits_E |f_n| = \limsup\{\int |f_n| - \int\limits_{E^c} |f_n|\} = \int |f| - \liminf \int\limits_{E^c} |f_n| \leq \int |f| - \int\limits_{E^c} |f|$ (by Fatou's Lemma)

$= \int\limits_E |f| \leq \liminf \int\limits_E |f_n|$ which implies that $\int\limits_E |f_n| \to \int\limits_E |f|$ for every measurable set E. Now proceed as in above proof and use the inequality $\int |f_n - f| \leq \int\limits_{A^c} |f| + \int\limits_{A^c} |f_n| + \int\limits_{A \backslash B} |f| + \int\limits_{A \backslash B} |f_n| + \int\limits_B |f_n - f|$].

Problem 574

Prove that L^p is uniformly convex if $1 < p < \infty$.

This follows immediately from Clarkson inequalities. Find details below.

Lemma 1

If $2 \le p < \infty$ then $\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \frac{|a|^p}{2} + \frac{|b|^p}{2}$ for any two complex numbers a and b.

(The inequality is false for $p = 1$ (and $p = 1.5$, and, perhaps, for every $p \in [1, 2)$): put $a = 2, b = 1$).

Assume that $p \ge 2$. We have to show that $\left|\frac{1+c}{2}\right|^p + \left|\frac{1-c}{2}\right|^p \le \frac{1}{2} + \frac{|c|^p}{2}$ for $|c| \le 1$. In other words we have to show that $\left|\frac{1+re^{it}}{2}\right|^p + \left|\frac{1-re^{it}}{2}\right|^p \le \frac{1}{2} + \frac{r^p}{2}$ for $0 \le r \le 1$ and $0 \le t \le 2\pi$. Let $f(t) = \left|1 + re^{it}\right|^p + \left|1 - re^{it}\right|^p = (1 + r^2 + 2r\cos t)^{p/2} + (1 + r^2 - 2r\cos t)^{p/2}$. Then $f'(t) \le 0$ for $0 \le t \le \frac{\pi}{2}$ (as seen by an explicit computation of the derivative). If we show that $f(0)(\equiv (1+r)^p + (1-r)^p)) \le 2^p(\frac{1}{2} + \frac{r^p}{2})$ it would follow that $f(t) \le 2^p(\frac{1}{2} + \frac{r^p}{2})$ for $0 \le t \le \frac{\pi}{2}$. This proves the inequality $\left|\frac{1+re^{it}}{2}\right|^p + \left|\frac{1-re^{it}}{2}\right|^p \le \frac{1}{2} + \frac{r^p}{2}$ for $0 \le r \le 1$ and $0 \le t \le \frac{\pi}{2}$. Since $\pi - t \in [0, \frac{\pi}{2}]$ for $\frac{\pi}{2} \le t \le \pi$ and $2\pi - t \in [0, \pi]$ for $\pi \le t \le 2\pi$ the inequality holds for all $t \in [0, 2\pi]$. It remains to show that $(1+r)^p + (1-r)^p \le 2^p(\frac{1}{2} + \frac{r^p}{2})$ for $0 \le r \le 1$ and $p \ge 2$.

For this we begin with the function $\psi(x) = (1+x)^{p-1} + (1-x)^{p-1} - 2^{p-1}$defined on $[0, 1]$. Its derivative is non-negative on $(0, 1)$ and since $\psi(1) = 0$ we get $\psi(x) \le 0$ for $x \in [0, 1]$. Next let $\phi(x) = (\frac{1}{x}+1)^p + (\frac{1}{x}-1)^p - 2^{p-1}(\frac{1}{x^p}+1)$. Then $\phi(1) = 0$ and $\phi'(x) = -\frac{p}{x^{p+1}}\psi(x) \le 0$. Hence ϕ is increasing, so $\phi(x) \le 0$ for $x \in (0, 1]$. This says $(\frac{1}{x}+1)^p + (\frac{1}{x}-1)^p \le 2^{p-1}(\frac{1}{x^p}+1)$ which is what we wanted to prove.

Lemma 2

If $1 < p \le 2$ and $q = \frac{p}{p-1}$then $|a+b|^q + |a-b|^q \le 2(|a|^p + |b|^p)^{\frac{1}{p-1}}$ for any two complex numbers a and b.

As in the proof of Lemma 1 we can reduce this to the following inequality:

$(1+x\ )^q + (1-x)^q \le 2(1+x^p)^{1/(p-1)}$ for $1 < p < 2$ and $0 < x < 1$. [Equality holds for $p = 2$ as well as for $x \in \{0, 1\}$]. This is equivalent to the following:

$(1 + \frac{1-t}{1+t})^q + (1 - \frac{1-t}{1+t})^q \le 2\{1 + (\frac{1-t}{1+t})^p\}^{1/(p-1)}$ for $0 < t < 1$. We shall show that

$(1+s^q)^{p-1} \le \frac{1}{2}\{(1+s\ )^p + (1-s)^p\}$ for $0 < s < 1$. It is easy to see that this last inequality gives the previous one. Consider $\frac{1}{2}\{(1+s\ )^p + (1-s)^p\} - (1+s^q)^{p-1}$. This function has a (uniformly convergent) power series expansion in s and the coefficient of s^k is

$\frac{p(p-1)\ldots(p-(2k-1))}{(2k)!}s^{2k} - \frac{p(p-1)\ldots(p-(2k-1))}{(2k-1)!}s^{q(2k-1)} - \frac{p(p-1)\ldots(p-2k)}{(2k)!}s^{2kq}$ which is $s^{2k}\frac{(2-p)(3-p)\ldots(2k-p)}{(2k-1)!}s^{2k}\{\frac{p(p-1)}{(2k)(2k-p)} - \frac{p-1}{2k}s^{q(2k-1)-2k} + \frac{p-1}{2k}s^{2kq-2k}\} = s^{2k}\frac{(2-p)(3-p)\ldots(2k-p)}{(2k-1)!}s^{2k}\{\frac{1-s^{\frac{2k-p}{p-1}}}{\frac{2k-p}{p-1}} - \frac{1-s^{\frac{2k}{p-1}}}{\frac{2k}{p-1}}\} \ge 0$ because $\frac{1-s^\alpha}{\alpha}$ is a decreasing function of α on $(0, \infty)$. This proves that each term in the power series is non-negative and so is the sum.

Lemma 3

If f is a non-negative function in L^p and g is a non-negative function in L^q where $0 < p < 1, q = \frac{p}{p-1}$ then $\int fg \geq (\int f^p)^{1/p}(\int g^q)^{1/q}$ provided $\int g^q > 0$.

Note that $g(x) > 0$ a.e.. Let $r = \frac{1}{p}$ and $\phi_1 = g^{-1/r}, \phi_2 = g^{1/r}f^{1/r}$. Then $\int f^p = \int \phi_1\phi_2 \leq (\int \phi_2^r)^{1/r}(\int \phi_1^s)^{1/s}$ where $s = \frac{r}{r-1} = \frac{1}{1-p}$. Hence $\int f^p \leq (\int fg)^p(\int g^{-\frac{p}{1-p}})^{1-p}$. This says $\int fg \geq (\int f^p)^{1/p}\frac{1}{(\int g^{-\frac{p}{1-p}})^{(1-p)/p}} = (\int f^p)^{1/p}(\int g^q)^{1/q}$.

Lemma 4

If $0 < p < 1$ and f and g are non-negative functions in L^p then $f + g \in L^p$ and $\|f + g\|_p \geq \|f\|_p + \|g\|_p$.

Since $|f + g|^p \leq 2^p\{|f|^p + |g|^p\}$ we get $f + g \in L^p$. Now $\int\{f + g\}^p = \int f\{f + g\}^{p-1} + \int g\{f + g\}^{p-1} \geq (\int f^p)^{1/p}(\int(f + g)^p)^{\frac{p-1}{p}} + (\int g^p)^{1/p}(\int(f + g)^p)^{\frac{p-1}{p}}$. Hence $(\int f^p)^{1/p} + (\int g^p)^{1/p} \leq (\int(f + g)^p)^{1+\frac{1-p}{p}} = (\int(f + g)^p)^{1/p}$.

Lemma 5

Clarkson inequality: $\left\|\frac{f+g}{2}\right\|_p^q + \left\|\frac{f-g}{2}\right\|_p^q \leq \{\frac{1}{2}\|f\|_p^q + \frac{1}{2}\|g\|_p^q\}^{1/(p-1)}$ if $1 < p < 2, q = \frac{p}{p-1}$ and $f, g \in L^p$.

We have $\left\|\left|\frac{f+g}{2}\right|^q\right\|_{p-1} + \left\|\left|\frac{f+g}{2}\right|^q\right\|_{p-1} \leq \left\|\left|\frac{f+g}{2}\right|^q + \left|\frac{f-g}{2}\right|^q\right\|_{p-1}$ by Lemma 4 (because $0 < p - 1 < 1$). Hence $\left\|\frac{f+g}{2}\right\|_p^q + \left\|\frac{f-g}{2}\right\|_p^q \leq \left\|\left|\frac{f+g}{2}\right|^q + \left|\frac{f-g}{2}\right|^q\right\|_{p-1} \leq \{\frac{1}{2}\|f\|_p^q + \frac{1}{2}\|g\|_p^q\}^{1/(p-1)}$ by Lemma 2.

Lemma 6

Carkson inequality for $2 \leq p < \infty$:

$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \leq \frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p \; \forall f, g \in L^{p.}$

This is immediate from Lemma 1.

Problem 575

Let $1 < p < \infty.$,$f_n \in L^p(n = 1, 2, ...), f \in L^p, f_n \to f$ weakly and $\|f_n\|_p \to \|f\|_p$. Show that $\|f_n - f\|_p \to 0$.

This is a general fact: if X is a uniformly convex normed linear space, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ then $\|x_n - x\| \to 0$. [L^p is uniformly convex by Problem 574]. For a proof of this general fact note that there is nothing to prove if $x = 0$. Otherwise $\|x_n\| > 0$ for n sufficiently large, say $n \geq n$, and we can define $\{y_n\}_{n \geq n_0}$ by $y_n = \frac{1}{\|x_n\|} x_n$. Let $y = \frac{1}{\|x\|} x$. Then $y_n \to y$ weakly. Claim:$\|y_n - y\| \to 0$. By uniform convexity it suffices to show that $\|y_n + y\| \to 2$. Suppose there is a sequence $n_j \to \infty$ such that $\left\|y_{n_j} + y\right\| < 2 - \delta$ $\forall j$ with $\delta > 0$. Choose $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(y) = 1$. Then $\left\|x^*(y_{n_j} + y)\right\| \leq \left\|y_{n_j} + y\right\| < 2 - \delta$. But $x^*(y_{n_j}) \to x^*(y) = 1$ so $x^*(y_{n_j} + y) \to 2$, a contradiction.

Problem 576

Show that $f_n \in L^1 (n = 1, 2, ...), f \in L^1, f_n \to f$ weakly and $\|f_n\|_1 \to \|f\|_1$ does not imply $\|f_n - f\|_1 \to 0$.

Let μ be the normalized Lebesgue measure on $[0, 2\pi], f_n(x) = 1 + \sin nx$ and $f(x) = 1$. Then $f_n \to f$ by Riemann Lebesgue Lemma. $\|f_n - f\|_1 = \frac{1}{2\pi} \int\limits_0^{2\pi} |\sin(nx)|\, dx = \frac{2}{\pi} \forall n$.

Problem 577

Let f be locally integrable on $\mathbb{R}$ and $\int |f(x)|^p\, dx = \infty$ where $1 \leq p < \infty$. Show that $\int f(x) g(x) dx = 0$ for all g in some dense subset of $L^q(\mathbb{R})$ where $q = \frac{p}{p-1}$.

Remark: of course, the set above is not dense if f is a non-zero element of $L^q(\mathbb{R})$.

Let M be the space of all functions g in $L^q(\mathbb{R})$ such that $\int fg$ exists and has the value 0. If this linear space is not dense in $L^q(\mathbb{R})$ then there exists $h \in L^p(\mathbb{R}) \backslash \{0\}$ such that $\int gh = 0$ for all $g \in M$. Let x and y be common Lebesgue points of f and h. Let $g = -(\int\limits_{y-\varepsilon}^{y+\varepsilon} f) I_{(x-\varepsilon, x+\varepsilon)} + (\int\limits_{x-\varepsilon}^{x+\varepsilon} f) I_{(y-\varepsilon, y+\varepsilon)}$.
Then $g \in L^q$ and $\int fg = 0$. Hence $0 = \int fh = -(\int\limits_{y-\varepsilon}^{y+\varepsilon} f)(\int\limits_{x-\varepsilon}^{x+\varepsilon} h) + (\int\limits_{y-\varepsilon}^{y+\varepsilon} h)(\int\limits_{x-\varepsilon}^{x+\varepsilon} f)$.
Dividing by $4\varepsilon^2$ and letting $\varepsilon \to 0$ we get $f(y)h(x) = h(y)f(x)$. It follows that $h = cf$ a.e. for some constant c. Since $h \neq 0$ it follows that $c \neq 0$ and since $h \in L^p(\mathbb{R})$ we get $f \in L^p(\mathbb{R})$, a contradiction.

Problem 578

If $\sum |a_n|^p = \infty$ where $1 \leq p < \infty$ show that the collection of all sequences $\{b_n\}$ in l^q where $q = \frac{p}{p-1}$ such that $\sum a_n b_n$ converges to 0 is dense in l^q.

This is similar to previous problem. Just replace intervals in the proof above by singletons.

Problem 579

If $1 \leq p < \infty$,$f_n \to f$ in L^p and $f_n \to g$ a.e. show that $f = g$ a.e.

We first observe that $\{f_n\}$ is bounded in the norm of L^p and hence $\int |g|^p \leq \liminf \int |f_n|^p < \infty$. If possible let $\mu\{x : |f(x) - g(x)| > \delta\} > 0$ for some $\delta > 0$. Let $0 < \varepsilon < \mu\{x : |f(x) - g(x)| > \delta\}$. By Egoroff's Theorem applied to the restriction of μ to $\{x : |f(x) - g(x)| > \delta\}$ (which is a finite measure) there exists $B \subseteq \{x : |f(x) - g(x)| > \delta\}$ such that $f_n \to g$ uniformly on B and $\mu(\{x : |f(x) - g(x)| > \delta\} \backslash B) < \varepsilon$. But then $\int\limits_E g = \lim \int\limits_E f_n = \int\limits_E f$ for every measurable set $E \subseteq B$. It follows that $f = g$ a.e. on B. Hence $\mu\{x : |f(x) - g(x)| > \delta\} = \mu(\{x : |f(x) - g(x)| > \delta\} \backslash B) < \varepsilon$ contradicting the choice of ε.

Problem 580

If $1 < p < \infty$ and $\{f_n\} \subseteq \{f \in L^p(m) : \|f\|_p \leq 1\}$ (where m is Lebesgue measure on $(0, 1)$) show that there is a subsequence of $\{f_n\}$ which converges weakly.

Remark: this is false for $p = 1$: let $f_n = nI_{(0, \frac{1}{n})}$. If $f_{n_j} \to h$ weakly then $\int hg = g(0)$ for any continuous function g on $[0, 1]$. In particular $\int xh(x)g(x) = 0$ for every continuous function g which implies $xh(x) = 0$ a.e.. However $\int h = \lim \int f_{n_j} = 1$.

Claim: if $\{f_n\}$ is a bounded sequence in $L^p(m)$ and $\lim\limits_{n\to\infty} \int\limits_0^x f_n$ exists for every x in a dense subset of $(0, 1)$ then $\{f_n\}$ converges weakly. To see this note that the unit ball of $L^p(m)$ is weakly compact and metrizable by Banach Aloaglu Theorem and separability of $L^q(m)$ where $q = \frac{p}{p-1}$. Hence any subsequence of $\{f_n\}$ has a further subsequence converging weakly to some function $g \in L^q(m)$. This implies that $\lim \int\limits_0^x g = \lim\limits_{n\to\infty} \int\limits_0^x f_n$, so g does not depend on the subsequence we started with. This proves the claim. Now given $\{f_n\}$ as in the statement there is, by a diagonal procedure, a subsequence $\{f_{n_j}\}$ such that $\lim\limits_{n\to\infty} \int\limits_0^x f_{n_j}$ exists for every rational number x. It follows by the claim that $\{f_{n_j}\}$ converges weakly in $L^p(m)$.

Problem 581

If X is a reflexive Banach space, Y is a normed linear space and $T : X \to Y$ is a bounded operator show that $\{Tx : \|x\| \leq 1\}$ is closed in Y.

This requires a theorem of Eberlein: the closed unit ball of X is weakly sequentially compact if X is reflexive. Thus, $\|x_n\| \leq 1, Tx_n \to y$ implies $x_{n_j} \to x$ weakly for some $n_j \uparrow \infty$ and some $x \in X$. Hence $Tx_{n_j} \to Tx$ weakly, so $y = Tx \in \{Tx : \|x\| \leq 1\}$ because $|x^*(x)| = \lim \left|x^*(x_{n_j})\right| \leq \|x^*\| \; \forall x^*$.

Problem 582

Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space H. Show that $\sum\limits_{n=1}^{\infty} x_n$ converges in the norm iff $\sum\limits_{n=1}^{\infty} < x_n, y >$ converges for every $y \in H$ iff $\sum\limits_{n=1}^{\infty} \|x_n\|^2 < \infty$.

Suppose $\sum\limits_{n=1}^{\infty} < x_n, y >$ converges $\forall y$. Define $T_n : H \to K$ (K being the scalar field) by $T_n(y) = \sum\limits_{k=1}^{n} < x_k, y > = < \sum\limits_{k=1}^{n} x_k, y >$. Then $\|T_n\| = \left\|\sum\limits_{k=1}^{n} x_k\right\| = \sqrt{\sum\limits_{k=1}^{n} \|x_k\|^2}$ by orthogonality. By Uniform Boundedness Principle $\sup\{\|T_n\| : n \geq 1\} < \infty$. Hence $\sum\limits_{n=1}^{\infty} \|x_n\|^2 < \infty$. Rest is elementary.

Problem 583

Show that the sequence $\{\frac{\sin(2\pi nx)}{|\sin(2\pi nx)|}\}$ converges to 0 weakly in $L^2(0,1)$.

We claim that $\sup\{\int\limits_a^b \frac{\sin x}{|\sin x|} dx : 0 < a < b < \infty\} < \infty$. For this observe that if $2\pi j \leq a < 2\pi(j+1)$ and $2\pi k \leq b < 2\pi(k+1)$ then $\int\limits_{2\pi l}^{2\pi(l+1)} \frac{\sin x}{|\sin x|} dx = 0$ for every l between $l+1$ and $k-1$. Hence $\int\limits_a^b \frac{\sin x}{|\sin x|} dx$ is the sum of integrals of $\frac{\sin x}{|\sin x|}$ over two intervals, each of which has length at most 2π. It follows that $\sup\{\int\limits_a^b \frac{\sin x}{|\sin x|}dx : 0 < a < b < \infty\} \leq 2\pi$. Now $\int\limits_a^b \frac{\sin(2\pi nx)}{|\sin(2\pi nx)|}dx = \frac{1}{2\pi n}\int\limits_{2\pi na}^{2\pi nb} \frac{\sin(y)}{|\sin(y)|}dx \to 0$. It follows that $\int\limits_0^1 \frac{\sin(2\pi nx)}{|\sin(2\pi nx)|}f(x)dx \to 0$ for every step function f. The fact that step fuctions form a dense subset of $L^2(0,1)$ completes the proof.

Problem 584

Let X and Y be Banach spaces, $T : X \to Y$ and $S^* : Y^* \to X^*$ linear and $y^*(T(x)) = S(y^*)(x)\ \forall x \in X, \forall y^* \in Y^*$. Show that T and S are bounded operators and $S = T^*$.

For $\|y^*\| \leq 1$ let $T_{y^*}(x) = y^*(Tx)(= S(y^*)(x))$. Then T_{y^*} is a continuous linear functional on X with $\|T_{y^*}\| \leq \|S(y^*)\|$. For fixed $x \in X$ we have $|T_{y^*}(x)| = |y^*(Tx)| \leq \|Tx\|$. By Uniform Boundedness Principle it follows that $\sup\{|T_{y^*}(x)| : \|y^*\| \leq 1, \|x\| \leq 1\} < \infty$ which implies $\sup\{\|Tx\| : \|x\| \leq 1\} < \infty$. Hence T is bounded. Similarly S is also bounded. By definition of adjoints $S = T^*$.

Problem 585

A book on Functional Analysis has an exercise which says that if A_1 and A_2 are self adjoint operators, $A_1 \geq A_2$ and $B \geq 0$ then $A_1 B \geq A_2 B$. Give a counterexample.

Let A_1 and B be the operators on $\mathbb{C}^2$ given by the matrices $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & -3 \\ -3 & 10 \end{pmatrix}$. Then $< A_1(x,y), (x,y) >= 2\{(x + \frac{1}{2}y)^2 + \frac{7}{4}y^2\}$ and $< B(x,y), (x,y) >= (x - 3y)^2 + y^2$. If $A_2 = 0$ then the hypothesis is satisfied. However, $A_1 B$ is given by the matrix $\begin{pmatrix} -1 & 4 \\ -11 & 37 \end{pmatrix}$ which is not positive definite since $< (A_1 B)(1,0), (1,0) >= -1 < 0$. (The conclusion would have been true if there was an added hypothsis that B commutes with $A_1 - A_2$).

Problem 586

Give an example of a non-empty closed set in a Hilbert space which has no element of minimal norm

Solution: $\{\alpha_n e_n : n \geq 1\}$ where $\alpha_n \to 1$ and $\{e_n\}$ is orhonormal.

Problem 587

Show that $T \in B(H)$ (H a Hilbert space), $T^2 = T$ and $\|T\| \leq 1$ imply $T^* = T$ (so T is an orthogonal projection).

First recall that $\|T^*\| = \|T\|$: $\|T^*\| = \sup\{< T^*x, y >: \|x\| \leq 1, \|y\| \leq 1\} = \sup\{< x, Ty >: \|x\| \leq 1, \|y\| \leq 1\} = \|T\|$.

If $Tx = x$ then $\|T^*x - x\|^2 = \|T^*x\|^2 + \|x\|^2 - 2\,\mathrm{Re} < T^*x, x >= \|T^*x\|^2 + \|x\|^2 - 2\,\mathrm{Re} < x, Tx >$

$\leq \|x\|^2 + \|x\|^2 - 2\,\|x\|^2 = 0$. Thus, $Tx = x$ implies $T^*x = x$. Since $T^2 = T$ this gives $T^*(Tx) = Tx\ \forall x$. Hence $T^*T = T$ and, taking adjoints on both sides, we get $T^*T = T^*$. Thus, $T^* = T$.

Problem 588

Let $\mathcal{M}$ be the space of all complex Borel measures on $[0,1]$. Is the set $\{\mu \in \mathcal{M} : \|\mu\| = 1\}$ a closed set in teh weak* topology of $\mathcal{M} \equiv (C[0,1])^*$? [$\|\mu\| = |\mu|([0,1])$ is the total variation norm of μ].

No! Let $d\mu_n = \frac{\pi}{2}\sin(2\pi nx)dx$ and $\mu = 0$. Then $\|\mu_n\| = 1 \forall n$ and $\mu_n \to \mu$ in the weak* topology (by Riemann Lebesgue Lemma).

Next four problems (from Bourbaki's "Integration I") are related to each other. [Some of these appeared earlier in Problems 357-358 but with slightly different proofs]

Problem 589

If $a_1, a_2, ..., a_N$ are distinct real numbers show that the functions $|x - a_i|, 1 \leq i \leq N$ are lineraly independent elements of $C[0,1]$.

Suppose $|x - a| = \sum_{j=1}^{k} b_j |x - a_j|$ with $b_j \in \mathbb{R}, 1 \leq j \leq k$ and $a, a_1, a_2..., a_N$ distinct. We show that at least one of the coefficients vanishes. The proof can then be completed using induction on k. For x large we have

$x - a = \sum_{j=1}^{k} b_j(x - a_j)$. Hence $\sum_{j=1}^{k} b_j = 1$. Without loss of generality we may suppose $a_1 < a_2 < ... < a_k$. We consider three cases:

a) $a_1 < a_2 < .... < a_{i-1} < a < a_i < ... < a_k$ for some i

b) $a < a_1 < a_2 < .... < a_k$

c) $a_1 < ... < a_k < a$

In case a) take $x \in (a_1, a_2)$ to get $a - x = \sum_{j=2}^{k} b_j(a_j - x) + b_1(x - a_1)$. This implies $\sum_{j=2}^{k} b_j = 1 + b_1$ so $b_1 = 0$.

In case b) let $x \in (a, a_1)$. We get $x - a = \sum_{j=1}^{k} b_j(a_j - x)$ which gives the contradiction $\sum_{j=1}^{k} b_j = -1$.

In case c) let $x \in (a_k, a)$. We have $a - x = \sum_{j=1}^{k} b_j(x - a_j)$ which again leads to the contradiction $\sum_{j=1}^{k} b_j = -1$.

Problem 590 [Continuation of Problem 589]

Show that we cannot express $|x - y|$ $(0 \leq x \leq 1, 0 \leq y \leq 1)$ as a finite sum of functions of the type $f(x)g(y)$ with f and g continuous.

Suppose $|x - y| = \sum_{i=1}^{k} f_i(x)g_i(y)$ for $0 \leq x \leq 1, 0 \leq y \leq 1$ with each f_i and each g_i continuous. Consider $k+1$ distinct numbers $y_1, y_2, ..., y_{k+1}$ in $[0,1]$. Then $|x - y_j| = \sum_{i=1}^{k} f_i(x)g_i(y_j)$ $\forall x$. By elementary linear algebra the system of equations $\sum_{j=1}^{k+1} \lambda_j g_i(y_j) = 0$ for $1 \leq i \leq k$ has a non-trivial solution. This gives $\sum_{j=1}^{k+1} \lambda_j |x - y_j| = \sum_{j=1}^{k+1} \lambda_j \sum_{i=1}^{k} f_i(x)g_i(y_j) = \sum_{i=1}^{k} \{\sum_{j=1}^{k+1} \lambda_j g_i(y_j)\} f_i(x) = 0$ $\forall x$ and this contradicts the linear independence of $\{|x - y_j| : 1 \leq j \leq k+1\}$ proved in Problem 589.

Problem 591 [Continuation of Problem 590]

If μ is a complex Borel measure on $[0,1]$ such that $\int |x - y| \, d\mu(x) = 0$ $\forall y \in [0,1]$ show that $\mu = 0$.

Remark: an equivalent statement is: $\{|x - y| : 0 \leq y \leq 1\}$ spans a dense subspace of $C[0,1]$.

A corollary is the following: if $X, X_1, X_2 ...$ are random variables with values in $[0,1]$ and $E|X_n - a| \to E|X - a|$ $\forall a$ then $X_n \to X$ weakly

Let $0 < y < 1$ and suppose $|\mu|\{y\} = 0$. For $|h|$ sufficiently small we have $0 = \frac{\int |x-y-h| d\mu(x) - \int |x-y| d\mu(x)}{h} = \int \phi(x,y,h) d\mu(x)$ where $\phi(x,y,h) = \frac{|x-y-h| - |x-y|}{h}$. Note that $|\phi(x,y,h)| \leq 1$ and $\phi(x,y,h) \to I_{[0,y]} - I_{(y,1]}$ a.e. $[\mu]$. By DCT we get $\mu[0,y] - \mu(y,1] = 0$. Note that if we put $y = 0$ in the hypothesis we get

$\int xd\mu(x) = 0$ and if we put $y = 1$ we get $\mu[0,1] - \int xd\mu(x) = 0$. It follows that $\mu[0,1] = 0$. Combining this with $\mu[0,y] = \mu(y,1]$ we get $\mu[0,y] = \mu(y,1] = 0$ $\forall y$ such that $|\mu|\{y\} = 0$. There are at most a countable number of points y such that $|\mu|\{y\} > 0$ and the function $\mu[0,y]$ is right continuous, so we get $\mu[0,y] = 0$ $\forall y \in (0,1)$. This implies that μ is concentrated on $\{1\}$; since $\int |x-y|\, d\delta_1(x) = |1-y| \neq 0$ (for any $y < 1$) we see that μ must be the zero measure.

Problem 592 [Continuation of Problem 591]

Consider the map $\mu \to \phi_\mu$ where $\phi_\mu(y) = \int |x-y|\, d\mu(x)$ from the space of complex Borel mesaures on $[0,1]$ into $C[0,1]$. Show that this is a one-to-one continuous linear map with dense range whose inverse is not continuous. Also show that the range is a proper subset of $C[0,1]$.

The second part follows from the first by Open Mapping Theorem. We have already shown (in Problem 591) that the linear map $\mu \to \phi_\mu$ is one-to-one. Note that $\left|\frac{1}{n} - y\right| \to |y|$ uniformly for $0 \le y \le 1$. Hence $\phi_{\delta_{\frac{1}{n}}} \to \phi_{\delta_0}$ in $C[0,1]$. Since $\left\|\delta_{\frac{1}{n}} - \delta_0\right\| = 2$ $\forall n$ we see that the inverse of above map is not continuous. To show that the range is dense we show that $\int \phi_\mu(y) d\nu(y) = 0$ $\forall \mu$ implies $\nu = 0$. Taking degenerate measures for μ we see that $\int |x-y|\, d\nu(y) = 0$ $\forall x$ which implies $\nu = 0$ by Porblem 591.

Problem 593

Let $0 < p < 1$ and X be the space $L^p(\mu)$ where μ is Lebesgue measure on $(0,1)$. Metrize X by $d(f,g) = \int |f-g|^p$. If V is any neighbourhood of 0 show that the convex hull of V equals X. Use this to show that there is no non-zero continuous linear functional on X.

Let $f \in X$. The function $x \to \int\limits_0^x |f(y)|^p\, dy$ is continuous. Hence there exists $a \in (0,1)$ such that $\int\limits_0^a |f(y)|^p\, dy = \frac{1}{2}\int\limits_0^1 |f(y)|^p\, dy$. Let $f_1 = 2fI_{(0,a)}$ and $f_2 = 2fI_{[a,1)}$. Then $\frac{f_1+f_2}{2} = f$. Also $\int |f_1|^p = 2^p \int\limits_0^a |f(y)|^p\, dy = 2^{p-1}\int\limits_0^1 |f(y)|^p\, dy$. Similarly, $\int |f_2|^p = 2^{p-1}\int\limits_0^1 |f(y)|^p\, dy$. We have expressed f as a convex combination of functions f_1 and f_2 such that $d(f_j, 0) = 2^{p-1}d(f,0)$. Repeating this we can express f as a convex combination of a finite number of elements whose distance from 0 is as small as we want. (Note that $p-1 < 0$). This proves the first part and the second part is immediate: $\{f : |x^*(f)| < 1\}$ is a convex neighbourhood of 0 for any continuous linear functional x^* on X and so $|x^*(nf)| < 1$ $\forall n\ \forall f$ so $x^* \equiv 0$.

Problem 594

Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and $1 \le p, q < \infty$. Find a necessary and suffcient condition that $f \circ g \in L^q(\mathbb{R})$ for every $g \in L^p(\mathbb{R})$.

The condition is $|f(x)| \le c\,|x|^{p/q}\ \forall x \in \mathbb{R}$ for some constant $c \in (0, \infty)$. Sufficiency is obvious. Necessity is proved by contradiction. Suppose $|f(x_n)| > n\,|x_n|^{p/q}\ \forall n$. Let $g = \sum\limits_{n=2}^{\infty} x_n I_{A_n}$ where $A_n's$ are disjoint Borel sets with $m(A_n) = \frac{1}{n^q |x_n|^p}$ if $q > 1$ and $m(A_n) = \frac{1}{n(\log n)^2 |x_n|^p}$ if $q = 1$. [If $x_n = 0$ then $m(A_n)$ can be arbitrary]. Then $\int |g|^p = \sum\limits_{n=2}^{\infty} \frac{1}{n^q}$ if $q > 1$ and $\int |g|^p = \sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ if $q = 1$. Thus $g \in L^p$. But $f \circ g \notin L^q(\mathbb{R})$ because $\int |f \circ g|^q = \sum\limits_{n=2}^{\infty} |f(x_n)|^q\, m(A_n) > \sum\limits_{n=2}^{\infty} n^q\, |x_n|^p\, m(A_n)$ which is $\sum\limits_{n=2}^{\infty} 1 = \infty$ if $q > 1$ and $\sum\limits_{n=2}^{\infty} \frac{1}{(\log n)^2} = \infty$.

Problem 595

Prove or disprove: the sequence $\{\sin nx\}$ converges to 0 *in measure* on the space $(0, 1)$ with Lebesgue measure.

If $\{\sin nx\}$ converges to 0 in measure then DCT tells us that $\int\limits_0^1 |\sin nx|\, dx \to 0$. [Indeed, every subsequence of $\{\sin nx\}$ has a further subsequence converging a.e. to 0]. However $\int\limits_0^1 |\sin nx|\, dx = \frac{1}{n} \int\limits_0^n |\sin y|\, dy \ge \frac{1}{n} \int\limits_0^{2k\pi} |\sin y|\, dy$ where $k = [\frac{n}{2\pi}]$. Hence $\int\limits_0^1 |\sin nx|\, dx \ge \frac{1}{n} \sum\limits_{j=1}^{k} \int\limits_{2(j-1)\pi}^{2j\pi} |\sin y|\, dy = \frac{1}{n} \sum\limits_{j=1}^{k} \int\limits_0^{2\pi} |\sin y|\, dy = \frac{4}{n}[\frac{n}{2\pi}] \to \frac{2}{\pi}$ as $n \to \infty$.

Problem 596

Let $f, g : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$. Show that $\det \begin{pmatrix} f(1) - f(0) & g(1) - g(0) \\ f'(c) & g'(c) \end{pmatrix} = 0$ for some $c \in (0, 1)$.

Apply Mean Value Theorem to the function $\det\begin{pmatrix} f(1)-f(0) & g(1)-g(0) \\ f(x)-f(0) & g(x)-g(0) \end{pmatrix}$.

Problem 597

For a function $f:(0,\infty)\to\mathbb{R}$ show that convexity of $xf(x)$ and $f(\frac{1}{x})$ are equivalent.

This is easy if we use the fact that a function $g:(0,\infty)\to\mathbb{R}$ is convex if and only if there exist sequences $\{a_n\},\{b_n\}$ such that $g(x)=\sup\{a_nx+b_n : n=1,2,...\}\ \forall x$.

Problem 598

Prove that a function $f:[a,b]\to\mathbb{R}$ is convex if and only if $\sup\{f(x)+ax : \alpha\leq x\leq\beta\}=\max\{f(\alpha)+a\alpha, f(\beta)+a\beta\}$ whenever $a\leq\alpha<\beta\leq b$.

If f is convex we can write $f(x)$as $\sup\{a_nx+b_n : n=1,2,...\}$ and affine maps satisfy the property in the statement which implies that f itself has this property. Conversely, suppose $\sup\{f(x)+ax : \alpha\leq x\leq\beta\}=\max\{f(\alpha)+a\alpha, f(\beta)+a\beta\}$ whenever $a\leq\alpha<\beta\leq b$. Let $a=-\frac{f(\beta)-f(\alpha)}{\beta-\alpha}$. Then $f(\alpha)+a\alpha=f(\beta)+a\beta$ and hence $f(x)+ax\leq f(\alpha)+a\alpha$ for $\alpha\leq x\leq\beta$. Thus $\frac{f(x)-f(\alpha)}{x-\alpha}\leq\frac{f(\beta)-f(\alpha)}{\beta-\alpha}$ for $\alpha<x<\beta$. Since α and β are arbitrary (subject to $a\leq\alpha<\beta\leq b$) this says $\frac{f(x)-f(\alpha)}{x-\alpha}\leq\frac{f(\beta)-f(\alpha)}{\beta-\alpha}$ whenever $\alpha<x<\beta$ with $\alpha,\beta,x\in[a,b]$. In other words, $f(x)\leq\frac{\beta-x}{\beta-\alpha}f(\alpha)+\frac{x-\alpha}{\beta-\alpha}f(\beta)$ whenever $\alpha<x<\beta$. Hence f is convex.

Problem 599

If f is convex on $[0,\infty)$ show that $f(x)-xf'(x+)$ is decreasing.

We have $f(x_2)-f(x_1)=\int\limits_{x_1}^{x_2}f'(x+)dx\leq f'(x_2+)(x_2-x_1)\leq x_2f'(x_2+)-x_1f'(x_1+)$ for $0\leq x_1<x_2$.

Problem 600

Let X and Y be jointly normal each having mean 0. Show that $\cos(\pi P\{XY<0\})=\rho(X,Y)$ (the correlation coefficient between X and Y.

Without loss of generality we may suppose that X and Y both have variance 1. We have to show that $EXY=\cos(\pi P\{XY<0\})$. Let $\alpha=EXY, U=X$ and $V=aX+bY$ where $a=-b\alpha$ and $b^2\{1-\alpha^2\}=1$. [This is not possible when $\alpha=\pm1$. However, if $\alpha=\pm1$ then, by the condition for equality in Holder's inequality) we get $Y=cX$ for some real number c which must be ±1 and the result is obvious in this case]. Note that $|\alpha|<1$ so we can take $b=\frac{1}{\sqrt{1-\alpha^2}}$ and $a=$

$-\frac{\alpha}{\sqrt{1-\alpha^2}}$. Now U and V are i.i.d. $N(0,1)$. Hence $P\{XY<0\} = P\{U\frac{V-aU}{b} < 0\} = P\{UV < aU^2\} = \int\int_{\{st<as^2\}} \frac{1}{2\pi}e^{-s^2/2}e^{-t^2/2}dsdt$. Using polar coordinates we get $P\{XY<0\} = \int_{\{\sin\theta\cos\theta<a\cos^2\theta\}}\int_0^\infty \frac{1}{2\pi}e^{-r^2/2}rdrd\theta = \frac{1}{2\pi}m\{\theta\in[0,2\pi] : \{\sin\theta\cos\theta < a\cos^2\theta\}\}$. Hence $\cos(\pi P\{XY<0\}) = \cos\{\frac{1}{2}m\{\theta\in[0,2\pi] : \{\sin\theta\cos\theta < a\cos^2\theta\}\})$. Splitting $[0,2\pi]$ into the parts with $\cos\theta>0$ and $\cos\theta<0$ we get $\frac{1}{2}m\{\theta\in[0,2\pi] : \{\sin\theta\cos\theta < a\cos^2\theta\}\} = m\{\theta\in[0,2\pi] : \{\tan\theta < a\}\}$. What remains is to show that $\cos(m\{\theta\in[0,2\pi] : \{\tan\theta < a\}\}) = \alpha$. Using the fact that $\cos(\pi P\{XY>0\}) = \cos(\pi-\pi P\{XY<0\}) = -\cos(\pi P\{XY<0\})$ we see that the original result does not change of we replace X by $-X$. Hence there is no loss of generality in assuming that $\alpha>0$. (in which case $a<0$). $m\{\theta\in[-\pi,\pi] : \{\tan\theta<a\} = \pi-2\theta_0$ where $\theta_0=\tan^{-1}(-a)$. [To see this split $[-\pi,\pi]$ into $[-\pi,-\pi/2],[-\pi/2,0],[0,\pi/2]$ and $[\pi/2,\pi]$'; there is no contribution from second and fourth intervals; contributions from the first and the third are each equal to $\frac{\pi}{2}-\theta_0]$. All that remains is to show that $\cos(\frac{\pi}{2}-\theta_0) = \alpha$ or $\sin\theta_0=\alpha$. This is easy: $\sin\theta_0 = \sqrt{1-\frac{1}{1+\tan^2\theta_0}} = -\frac{1}{\sqrt{1+a^2}} = \alpha$.

Problem 601

Let $f, f_1, f_2, ...$ be convex functions on $[0,1]$ such that $f_n(x)\to f(x)\ \forall x\in[0,1]$. Show that $f_n\to f$ uniformly on $[0,1]$.

Remark: the argument below shows that if we $f_n's$ are convex, $\{f_n(0)\}$ is bounded above, $\{f_n(1)\}$ is bounded above and $\{f_n(\frac{1}{2})\}$ is bounded below then $\{f_n\}$ is equicontinous.

Fix $0<c<1$. We have $f_n(c) = f_n(0)+\int_0^c g_n(t)dt$ where $g_n(t)=f_n'(t+)$. Since g_n is increasing we have $f_n(c)\le f_n(0)+cg_n(c)$. Hence there exists a positive number M such that $g_n(c)>-M\ \forall n$. On $[c,1]$ we can write $f_n(x) = \sup\{a_{nj}x+b_{nj} : j\ge1\}$ where each a_{nj} is $g_n(t)$ for some t in $[c,1]$. It follows that $a_{nj}>-M\ \forall n,j$. Now $\sup\{b_{nj} : j\ge1\}$ and $\sup\{a_{nj}+b_{nj} : j\ge1\}$ are both finite because $\{f_n(0)\}$ and $\{f_n(1)\}$ are bounded. It follows that $\sup\{b_{nj} : j\ge1\}<\infty$ and $\sup\{a_{nj} : j\ge1\}<\infty$. Let $x,y\in[c,1]$. Then $a_{nj}x+b_{nj} = \{a_{nj}y+b_{nj}\}+a_{nj}(x-y)\le f_n(x)+M_1|x-y|$ where $M_1=\sup\{|a_{nj}| : n,j\ge1\}$. Taking supremum over j we get $f_n(x)\le f_n(y)+M_1|x-y|$. It follows from this that $|f_n(x)-f_n(y)|\le M_1|x-y|\ \forall x,y,\forall n$. We conclude that $\{f_n\}$ is equicontinuous and $f_n\to f$ uniformly on $[c,1]$. Applying this result to $f_n(1-x)$ and $f(1-x)$ we see that $f_n\to f$ uniformly on $[0,1-c]$. Take $c=\frac{1}{2}$ to complete the proof.

Problem 602

Let f be convex in $[a,b]$. Show that there exists a sequence of C^∞ convex functions converging uniformly to f.

Let $g(x) = \begin{cases} f'(x+) \text{ if } a \le x < b \\ f'(b-) \text{ if } b \le x < \infty \\ f'(a+) \text{ if } x < a \end{cases}$. Replacing f by $f(x) - f'(a+)x$ we may suppose $f'(a+) = 0$. Note that g is a bounded increasing function on $\mathbb{R}$. Let $\phi(x) = f(a) + \int\limits_{-\infty}^{x} g(t)dt$. Then ϕ is convex on $\mathbb{R}$ and $|\phi(x)| \le a + \beta |x|$ $(x \in \mathbb{R})$ for some $\alpha, \beta > 0$. Let $\phi_n(x) = \sqrt{\frac{n}{2\pi}} \int \phi(x-y) e^{-\frac{n}{2}y^2} dy$. Clearly, ϕ_n is well defined and convex on $\mathbb{R}$. We have $|\phi_n(x) - \phi(x)| \le \sqrt{\frac{n}{2\pi}} \int |\phi(x-y) - \phi(x)| e^{-\frac{n}{2}y^2} dy$. Let $\varepsilon > 0$. Since ϕ is (Lipschitz, hence) uniformly continuous we can find $\delta > 0$ such that $|\phi(x-y) - \phi(x)| < \varepsilon$ if $|y| < \delta$. Hence $\sqrt{\frac{n}{2\pi}} \int\limits_{\{|y|<\delta\}} |\phi(x-y) - \phi(x)| e^{-\frac{n}{2}y^2} dy < \varepsilon$. Now $\sqrt{\frac{n}{2\pi}} \int\limits_{\{|y|\ge\delta\}} |\phi(x-y) - \phi(x)| e^{-\frac{n}{2}y^2} dy \le \sqrt{\frac{n}{2\pi}} \int\limits_{\{|y|\ge\delta\}} [a + \beta |x| + a + \beta |x-y|] e^{-\frac{n}{2}y^2} dy$

$\le \sqrt{\frac{n}{2\pi}} \int\limits_{\{|y|\ge\delta\}} [a_1 + \beta_1 |y|] e^{-\frac{n}{2}y^2} dy$ for some finite constants if $x \in [a,b]$. Since $\sqrt{\frac{n}{2\pi}} \int\limits_{\{|y|\ge\delta\}} [a_1 + \beta_1 |y|] e^{-\frac{n}{2}y^2} dy = \sqrt{\frac{1}{2\pi}} \int\limits_{\{|u|\ge\sqrt{n}\delta\}} [a_1 + \beta_1 \left|\frac{u}{\sqrt{n}}\right|] e^{-\frac{1}{2}u^2} du \to 0$ as $n \to \infty$ we see that $\phi_n \to \phi$ uniformly on $[a,b]$. But $\phi = f$ on $[a,b]$. Hence, it remains only to show that each ϕ_n is a C^∞ function. Note that $\phi_n(x) = \sqrt{\frac{n}{2\pi}} \int \phi(y) e^{-\frac{n}{2}(x-y)^2} dy$. Repeated use of DCT together with the estimate $|\phi(x)| \le a + \beta |x|$ shows that ϕ_n is indeed a C^∞ function.

Problem 603.

In previous problem show that $\phi_n's$ can be modified to be a decreasing/increasing sequence.

Given $\varepsilon > 0$ there exist a C^∞ convex functions ψ_n such that $|f(x) - \psi_n(x)| < \varepsilon/2^n$ $\forall x \in [a,b], \forall n$. Let $\phi_n(x) = \psi_n(x) + \varepsilon_n$ where $\varepsilon_1 = \sum\limits_{j=1}^{\infty} \left\|\psi_n - \psi_{n+1}\right\|$ and $\varepsilon_n - \varepsilon_{n+1} = \left\|\psi_n - \psi_{n+1}\right\| < \frac{2\varepsilon}{2^{n+1}}$ (the norm is, of course, the supremum norm). Then $\{\varepsilon_n\}$ strictly decreases to 0. Clearly, ϕ_n is convex and C^∞. Also $\phi_n \to f$ uniformly and $\phi_{n+1} = \psi_{n+1} + \varepsilon_n - \left\|\psi_n - \psi_{n+1}\right\| \le \psi_n + \varepsilon_n = \phi_n$ so $\{\phi_n\}$ decreases to f uniformly. A similar argument can be given to produce C^∞ convex functions increasing uniformly to f.

Problem 604

Give an example of a sequence of Lipschitz functions on $[0,1]$ converging uniformly whose limit is not Lipschitz.

$f_n(x) = \min\{nx, \sqrt{x}\}$ with limit $\sqrt{x}$.

Problem 605

Prove or disprove: if $f, f_1, f_2, ...$ are C^∞ functions on $\mathbb{R}$ with compact support and $f_n \to f$ uniformly on $\mathbb{R}$ then $f_n \to f$ in $L^1(\mathbb{R})$.

False: there exist C^∞ functions f_n with compact support such that $f_n = \frac{1}{n}$ for $-n \le x \le n$ and $0 \le f_n(x) \le \frac{1}{n}$ $\forall n, \forall x$. Clearly, $f_n \to 0$ uniformly and $\int |f_n - 0| \ge \int\limits_{-n}^{n} f_n = \frac{2n}{n} = 2$.

Problem 606

Prove the following:

a) If μ is a positive finite measure, f is a complex valued integrable function on $\mathbb{R}$ with $\left|\int f d\mu\right| = \int |f|\, d\mu$ then there exists a real constant a and a non-negative $L^1(\mu)$ function g such that $f = e^{ia} g$ a.e. $[\mu]$.

b) If μ is a complex Borel measure on $\mathbb{R}$, f is a complex valued integrable function on $\mathbb{R}$ with $\left|\int f d\mu\right| = \int |f|\, d\,|\mu|$ then there exists a real constant a such that $f\phi = e^{ia}\,|f|$ a.e. $[|\mu|]$ where $\phi = \frac{d\mu}{d|\mu|}$.

c) If μ is a complex Borel measure on $\mathbb{R}$ with $\left|\int f d\mu\right| = \int |f|\, d\,|\mu|$ for every $f \in C_c(\mathbb{R})$ then $\mu = c\delta_\alpha$ for some real number α and some $c \in S^1$.

a) Let $\int f d\mu = re^{ia}$ with $r \ge 0$ and a real. Then $\int |f|\, d\mu = r = \int e^{-ia} f d\mu$ so $\int \{|f| - \operatorname{Re} e^{-ia} f\} d\mu = 0$. Since $|f| - \operatorname{Re} e^{-ia} f \ge 0$ we get $|f| - \operatorname{Re} e^{-ia} f = 0$ a.e. $[\mu]$. This implies $\operatorname{Im} e^{-ia} f = 0$ a.e. $[\mu]$ and hence $f = e^{ia}\,|f|$ a.e. $[\mu]$.

b). We have $\left|\int f\phi d\nu\right| = \int |f|\, d\nu$ where $\nu = |\mu|$. By part a) and the fact that $|\phi| = 1$ a.e. $[\nu]$ we get $f\phi = e^{ia} g$ and $g = |f|$ necessarily so $f\phi = e^{ia}\,|f|$ a.e. $[\nu]$.

c) Since $C_c(\mathbb{R})$ is dense in $L^1(|\mu|)$ the equation $\left|\int f d\mu\right| = \int |f|\, d\,|\mu|$ holds for every $f \in L^1(|\mu|)$. Taking $f = 1$ we get (from part b)) $\phi = e^{ia}$ a.e. $[\nu]$. It follows that $d\mu = cd\,|\mu|$ where $c = e^{ia}$. We now have $\left|\int f d\nu\right| = \int |f|\, d\nu$ for every $f \in L^1(\nu)$ and, by part a), $f = e^{ia}\,|f|$ a.e. $[\nu]$ for every $f \in L^1(\nu)$ (the real constant a possibly depending on f). Let A and B be disjoint Borel sets. Taking f to be $I_A - I_B$ we get $\nu(A) = 0$ or $\nu(B) = 0$. The only positive finite Borel measures ν on $\mathbb{R}$ such that $\nu(A) = 0$ or $\nu(B) = 0$ whenever A and B are disjoint Borel sets are the degenerate measures: for each positive integer n there is a unique integer i_n such that $\nu([\frac{i_n - 1}{2^n}, \frac{i_n}{2^n})) > 0$. Since $[\frac{i_n - 1}{2^n}, \frac{i_n}{2^n})$ and $[\frac{i_{n+1} - 1}{2^{n+1}}, \frac{i_{n+1}}{2^{n+1}})$ must intersect we must have $[\frac{i_{n+1} - 1}{2^{n+1}}, \frac{i_{n+1}}{2^{n+1}}) \subseteq [\frac{i_n - 1}{2^n}, \frac{i_n}{2^n})$. Hence the intervals $[\frac{i_n-1}{2^n}, \frac{i_n}{2^n}], n = 1, 2, ...$ form a decreasing sequence of closed sets with diameters tending to 0. If α is their common point then $\nu = \delta_\alpha$ because any point $x \neq \alpha$ belongs to a dyadic interval I with $\nu(I) = 0$. We have proved that $\mu = c\delta_\alpha$ where $|c| = 1$ and α is real.

Problem 607

Let $f_n : [0, 2\pi] \to \mathbb{R}$ satisfy the inequalities $|f_n(x) - f_n(y)| \leq M|x-y|$ $\forall x, y \in [0, 2\pi], \forall n$ with M independent of n. Show that $\int_0^{2\pi} f_n(x)\{\sin nx\}dx \to 0$ as $n \to \infty$. If $\{f_n\}$ is uniformly bounded prove that $\int_0^{2\pi} \phi(x)f_n(x)\{\sin nx\}dx \to 0$ as $n \to \infty$ for every $\phi \in L^1([0, 2\pi])$.

Each f_n is absolutely continuous, hence differentiable a.e.. Let $g_n(x) = \frac{f_n(x)\cos(nx)}{2\pi n}$. Then g_n is absolutely continuous (because product of two absolutely continuous functions on $[0, 2\pi]$ is absolutely continuous). Hence $0 = g_n(2\pi) - g_n(0) = \int_0^{2\pi} g_n'(t)dt = \int_0^{2\pi} \frac{f_n'(t)\cos(nt) - nf_n(t)\sin(nt)}{2\pi n}dt$. Note that $|f_n'| \leq M$ a.e. by hypothesis and hence $\int_0^{2\pi} \frac{f_n'(t)\cos(nt)}{2\pi n}dt \to 0$. It follows that $\int_0^{2\pi} \frac{nf_n(t)\sin(nt)}{2\pi n}dt \to 0$, as required. For the second part let $\varepsilon > 0$ and choose a continuously differentiable fucntion ψ such that $\int|\phi - \psi| < \varepsilon$. Then $\left|\int_0^{2\pi} \phi(x)f_n(x)\{\sin nx\}dx - \int_0^{2\pi} \psi(x)f_n(x)\{\sin nx\}dx\right| \leq \varepsilon \sup \|f_n\|_\infty$. Since $|\psi(x)f_n(x) - \psi(y)f_n(y)| \leq |\psi(x)f_n(x) - \psi(y)f_n(x)| + |\psi(y)f_n(x) - \psi(y)f_n(y)| \leq M_1|x-y| \, \forall x, y \in [0, 2\pi], \forall n$ with M_1independent of n ($M_1 = \{(\sup \|f_n\|_\infty) \|\psi'\|_\infty + \|\psi\|_\infty\}M$ will do) we can apply the first case with $\{f_n\}$ replaced by $\{\psi f_n\}$ to $\int_0^{2\pi} \psi(x)f_n(x)\{\sin nx\}dx \to 0$. It follows that $\left|\int_0^{2\pi} \phi(x)f_n(x)\{\sin nx\}dx\right| < \varepsilon \sup \|f_n\|_\infty + \left|\int_0^{2\pi} \psi(x)f_n(x)\{\sin nx\}dx\right| < \varepsilon(1 + \sup \|f_n\|_\infty)$ for n sufficently large.

Alternative proof of the first part: by Arzela - Ascoli Theorem there is a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ converging uniformly to a continuous function f. Now $\left|\int_0^{2\pi} f_{k_j}(x)\{\sin k_jx\}dx - \int_0^{2\pi} f(x)\{\sin k_jx\}dx\right| \to 0$ as $j \to \infty$. Since

$\int_0^{2\pi} f(x)\{\sin k_j x\}dx \to 0$ too we conclude that $\int_0^{2\pi} f_{k_j}(x)\{\sin k_j x\}dx \to 0$. Arguing with subsequences we conclude that $\int_0^{2\pi} f_n(x)\{\sin nx\}dx \to 0$ as $n \to \infty$.

Problem 608

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. If G_f is closed show that f is continuous. More generally show that if $f : \mathbb{R} \to \mathbb{R}$ has closed graph then f is continuous at x if and only if it is bounded in some neighbourhood of x. Conclude that the set of points of continuity of a function with closed graph is necessarily open.

Remark: in general the set of points of continuity of any function $f : \mathbb{R} \to \mathbb{R}$ is a G_δ.

First part: if it is not true that $\lim_{y \to x} f(y) = f(x)$ then there exists a sequence $\{x_n\}$ converging to x such that $\{f(x_n)\}$ does not converge to $f(x)$ and hence there is a limit point y of this sequence with $y \neq f(x)$. Let
$f(x_{n_k}) \to y$. Since $\{(x_{n_k}, f(x_{n_k})\} \subseteq G_f$ and $\{(x_{n_k}, f(x_{n_k})\} \to (x, y)$ in $\mathbb{R}^2$ we must have $y = f(x)$, a contradiction. For the second part repeat above argumet under the assumtion that f is bounded in some neighborhood of x. The last part is obvious.

Problem 609

Show that there is a function $f : \mathbb{R} \to \mathbb{R}$ which has closed graph and uncountably many discontinuities.

Remark: such a function cannot be bounded by Problem 608.

Let C be the Cantor set, $f(x) = \begin{cases} \frac{1}{d(x,C)} \text{ if } x \notin C \\ 0 \text{ if } x \in C \end{cases}$. If $x_n \to x$ and $f(x_n) \to y$ with $x_n \notin C\ \forall n$ then $\frac{1}{d(x_n,C)} \to y$ and hence $\{d(x_n, C)\}$ does not tend to 0. Since $d(x_n, C) \to d(x, C)$ this implies $x \notin C$. Hence $\frac{1}{d(x_n,C)} \to \frac{1}{d(x,C)}$ and so $y = \lim f(x_n) = \lim \frac{1}{d(x_n,C)} = \frac{1}{d(x,C)} = f(x)$. If $x_n \in C\ \forall n$ then $y = \lim f(x_n) = 0, x \in C$ and $f(x) = 0$ so $y = f(x)$. Either $x_n \notin C$ along a subsequence or $x_n \in C$ along a subsequence. It follows that $y = f(x)$ in all cases, so G_f is closed. Claim: f is not continuous at any point of C: if $x \in C$ there is a sequence $\{x_n\}$ in $\mathbb{R} \backslash C$ converging to x. If f is continuous at x then $\frac{1}{d(x_n,C)} \to 0$ and $d(x_n, C) \to \infty$ which is absurd.

Problem 610

Show that if a function $f : \mathbb{R} \to \mathbb{R}$ has closed graph then the points of discontinuity is a closed set with empty interior; conversely any closed set with empty interior is the set of discontinuity points of a function with closed graph.

Remark: if $f : \mathbb{R} \to \mathbb{R}$ has closed graph then it has at least one point of continuity since the set points of discontinuity cannot be $\mathbb{R}$.

The second part follows by replacing the Cantor set C in previous problem by the given closed set with empty interior. For the first part note that the set D of discontinuity points is closed by the last part of Problem 608. It remains to show that D has no interior. Suppose $a < b$ and $[a, b] \subseteq D$. Then $[a, b] \subseteq \bigcup\limits_n \{x \in [a, b] : |f(x)| \leq n\}$. Claim:$\{x \in [a, b] : |f(x)| \leq n\}$ is closed for each n. Indeed, if $|f(x_k)| \leq n$ and $x_k \to x$ then, for any limit point y of $\{f(x_k)\}$ there is a subsequence $\{f(x_{k_j})\}$ converging to y. Since G_f is closed we get $(x, y) \in G_f$ so $y = f(x)$. We have proved that $f(x_k) \to f(x)$ so $|f(x)| \leq n$, proving the claim. By Baire Category Theorem we conclude that $\{x \in [a, b] : |f(x)| \leq n\}$ has non-empty interior for some n. But then f is continuous at any point of the interior (because f is bounded in a neighbourhood of such a point; see second part of Problem 608). We have arrived at a contradiction since $[a, b] \subseteq D$. The proof is complete.

Problem 611

Give an example of a σ - finite Borel measure μ on $\mathbb{R}$ such that $\mu([a, b]) = \infty$ whenever $a < b$.

Let $\{r_n\}$ be an enumeration of rationals, $f(x) = \frac{1}{\sqrt{x}} I_{(0,1)}(x)$ and $g(x) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} f(x + r_n)$. Note that $\int \sum\limits_{n=1}^{\infty} \frac{1}{2^n} f(x + r_n) dx = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \int f(x + r_n) dx = \sum\limits_{n=1}^{\infty} \frac{1}{2^{n-1}} < \infty$ and hence $g(x) < \infty$ a.e.. Let $\mu(E) = \int\limits_E g^2(x) dx$. Since $\mu\{x : |x| \leq n, g(x) \leq n\} < \infty \ \forall n$ it follows that μ is $\sigma-$ finite. If $a < b$ then $\mu([a, b]) = \int\limits_a^b (\sum\limits_{n=1}^{\infty} \frac{1}{2^n} f(x + r_n))^2 dx \geq \frac{1}{2^n} \int\limits_a^b (f(x + r_n))^2 dx = \frac{1}{2^n} \int\limits_a^b \frac{1}{x + r_n} I_{0 < x + r_n < 1}(x) dx = \frac{1}{2^n} \int\limits_{a+r_n}^{b+r_n} \frac{1}{y} I_{0 < y < 1}(x) dy = \infty$ if n is chosen such that $a + r_n < 0 < b + r_n$ or $r_n \in (-b, -a)$.

Problem 612

Let $f : \mathbb{R} \to \mathbb{C}$ be mesurable and bounded on compact sets with $f(x + y) = f(x) f(y) \ \forall x, y$ and assume that f is not identically 0. Prove that $f(x) \equiv e^{\alpha x}$ for some complex number α. If f is bounded show that f necessarily take values in S^1 and $f(x) \equiv e^{i\beta x}$ for some real number β. Conclude that if $f: S^1 \to S^1$ is a mesurable homomorphism then $f(x) \equiv z^n$ for some integer n.

[See also Problem 280 and Problem 613]

Let $g(x) = \int\limits_0^x f(t)dt$. Then $g(x+y) - g(x) = \int\limits_x^{x+y} f(t)dt = \int\limits_0^y f(s+x)ds = f(x)\int\limits_0^y f(s)ds = f(x)g(y)$. If $g(y) \neq 0$ this gives $f(x) = \frac{g(x+y)-g(x)}{g(y)}$ $\forall x$. Since g is absolutely continuous on finite intervals so is f by this formula. [$g \equiv 0$ implies $f \equiv 0$]. In particular g is continuously differentiable and so is f by above formula. Now $f'(x+y) = f(x)f'(y)$. Put $y = 0$ and solve the equation $f'(x) = f(x)f'(0)$ to get $f(x) = e^{f'(0)x}$. [$f^2(0) = f(0)$ so $f(0) = 0$ or 1. If $f(0) = 0$ then $f(x) = f(0)f(x) = 0$ $\forall x$ so $f(0)$ must be 1]. We have proved the first part with $\alpha = f'(0)$. Suppose f is also bounded. Since $|e^{\alpha x}| = e^{(\operatorname{Re}\alpha)x}$ we must have $\operatorname{Re}\alpha = 0$ proving that $f(x) \equiv e^{i\beta x}$ for some real number β. Finally if $f: S^1 \to S^1$ is a mesurable homomorphism and $h(t) = f(e^{i2\pi t})$ then $h(t) \equiv e^{i\beta t}$ for some real number β so $f(e^{2\pi i t}) = e^{i\beta t}$. The fact that $f(1) = 1$ forces $h(1)$ to be 1 and hence $\beta/2\pi$ is an integer n. Hence $f(z) = z^n$.

Problem 613

Let $f: \mathbb{R} \to \mathbb{C}$ be mesurable with $f(x+y) = f(x)f(y)$ $\forall x, y$ and assume that f is not identically 0. Prove that $f(x) \equiv e^{\alpha x}$ for some complex number α. Prove that all measurable homomorphisms of S^1 are of the type $z \to z^n$ for some integer n.

[Local boundedness has been dropped from Problem 612].

Let $f: \mathbb{R} \to \mathbb{C}$ be mesurable with $f(x+y) = f(x)f(y)$ $\forall x, y$ and assume that f is not identically 0. Let $\phi(x) = |f(x)|$. If $f(x) = 0$ for some x then $f(y) = f(x)f(y-x) = 0$ $\forall y$ so $\phi(x) > 0$ for all x. Now $\log\phi(x)$ is an additive measurable map of $\mathbb{R}$ into itself and hence $\log\phi(t) \equiv ct$ for some real number c. Now consider the function $f_1(t) = \frac{f(t)}{\phi(t)}$. Problem 612 can be applied to this function and we get $f_1(t) = e^{i\beta t}$ for some real number β. Thus $f(t) = f_1(t)\phi(t) = e^{i\beta t + ct}$. The second part is proved as in Problem612.

Problem 614

Prove Vitali - Hahn - Saks Theorem: if $\{\mu_n\}$ is a sequence of complex measures on $(\Omega, \mathcal{F})$ and $\mu(A) \equiv \lim\limits_{n\to\infty} \mu_n(A)$ exists for each $A \in \mathcal{F}$ then μ is a complex measure.

Remark: Problem 335 shows that the result fails of we replace complex measures by positive measures.

Let $\lambda(A) = \sum_{n=1}^{\infty} \frac{|\mu_n|(A)}{2^n\{1+|\mu_n|(\Omega)\}}$. Then λ is a positive finite measure and each μ_n is absolutely continuous w.r.t. λ. Let S be the set of all $\{0,1\}$ valued functions in $L^1(\lambda) : S = \{I_A : A \in \mathcal{F}\}$. Then S is closed in L^1 and hence it is a complete metric space. Let $\varepsilon > 0$. Then $S = \bigcup_{k=1}^{\infty} \bigcap_{n,m \geq k} \{I_A : |\mu_n(A) - \mu_m(A)| \leq \varepsilon\}$. By Baire Category Theorem there exists $k \in \mathbb{N}, A \in \mathcal{F}$ and $r > 0$ such that $\|I_B - I_A\|_1 < r$ implies $|\mu_n(B) - \mu_m(B)| \leq \varepsilon$ whenever $n, m \geq k$. Hence $\|I_B - I_A\|_1 < r$ implies $|\mu_n(B) - \mu(B)| \leq \varepsilon$ whenever $n \geq k$. If $\lambda(E) < r$ then $\|I_{A\cup E} - I_A\|_1 < r$ and $\|I_{A\backslash E} - I_A\|_1 < r$. Hence $|\mu_n(A \cup E) - \mu(A \cup E)| \leq \varepsilon$ and $|\mu_n(A\backslash E) - \mu(A\backslash E)| \leq \varepsilon$ whenever $n \geq k$. Since $\mu(E) = \mu(A\cup E) - \mu(A\backslash E)$ and $\mu_n(E) = \mu_n(A \cup E) - \mu_n(A\backslash E)$ we get $|\mu_n(E) - \mu(E)| \leq 2\varepsilon$ whenever $n \geq k$. In particular this holds for $n = k$ and the fact that $\mu_k << \lambda$ shows that $\mu(E) \to 0$ as $\lambda(E) \to 0$. It follows easily from this that μ is countably additive.

Problem 615

Let $a_n > 0\ \forall n$. Show that if $\sum_{n=1}^{\infty} a_n \sin nx$ is a Fourier series the $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$.

Suppose there is a function f in $L^1[0, 2\pi]$ such that $\sum_{n=1}^{\infty} a_n \sin nx$ is the Fourier series of f. Since $\sin nx = \frac{e^{inx} - e^{-inx}}{2i}$ we get $\hat{f}(n) = \frac{a_n}{2i}$ if $n \geq 1, \hat{f}(n) = -\frac{a_{-n}}{2i}$ if $n \leq -1$ and 0 if $n = 0$. Let $g(x) = \int_0^x f(t)dt$. Then $\hat{g}(n) = -\frac{a_n}{2n}$ if $n \geq 1, \hat{g}(n) = \frac{a_{-n}}{2n}$ if $n \leq -1$. Let $\{F_N\}$ be the Fejer sequence. Then $F_N * g \to g$ uniformly by Fejer's Theorem. In particular this gives $\sum_{n=-N}^{N} (1 - \frac{|n|}{N+1})\hat{g}(n) \to g(0)$ from which the conclusion follows easily.

Problem 616

If f is upper semi-continuous on a complete metric space X show that it is continuous at all points except those on a set of first category.

Let $\varepsilon > 0$ and $A_\varepsilon = \{x :$ there exists sequences $\{x_n\}, \{y_n\}$ converging to x with $f(x_n) - f(y_n) > \varepsilon\ \forall n\}$. [A_ε is the set of points at which the oscillation of f exceeds ε].

We claim that each A_ε is a closed set with empty interior and f is continuous on the complement of $\bigcup_{n=1}^{\infty} A_{1/n}$. Let $\{u_j\} \subseteq A_\varepsilon$ and $u_j \to u$. If $u \notin A_\varepsilon$ then there is a ball $B(u, \delta)$ such that $f(x) - f(y) \leq \varepsilon\ \forall x, y \in B(u, \delta)$. If j is so large that $d(u_j,u) < \delta/2$ then $u_j \in A_\varepsilon$ and hence there exist ξ,ζ such that $d(\xi,u_j) < \delta/2, d(\xi,u_j) < \delta/2$ and $f(\xi)-f(\zeta) > \varepsilon$. This is a contradiction because $\xi,\zeta \in B(u,\delta)$. Hence A_ε is closed. If possible let A_ε have an interior point u. Since f is upper semi-continuous $f(w) < f(u)+\frac{\varepsilon}{2}$ whenever $d(w,u)$ is sufficiently small. Let $\{x_n\},\{y_n\}$ converge to u with $f(x_n)-f(y_n) > \varepsilon \ \forall n$. But then $f(y_n) < f(x_n)-\varepsilon < f(u)-\varepsilon/2$ for n sufficiently large. This proves that there are points z in the interior of A_ε which are arbitrarily close to u satisfying the inequality $f(z) < f(u)-\varepsilon/2$. Repeating this argument we get points $z_1,z_2,...$ such that $f(z_n) < f(u)-\frac{n\varepsilon}{2}$. This would be a contradiction if f is bounded below. In particular we have proved that the upper semi-continuous function e^f is continuous. Hence f itself is continuous. Finally if $x \notin \bigcup\limits_{n=1}^{\infty} A_{1/n}$ then for any N, $f(v)-f(u) \leq \frac{1}{N}$ whenever u and v are sufficiently close to x proving that f is continuous at x.

Problem 617

Let C be an unbounded convex set in $\mathbb{R}^k$. Show that C contains a ray, i.e. it contains $\{a+tx : t \geq 0\}$ for some $a,x \in \mathbb{R}^k, x \neq 0$.

The proof can be reduced to the case when C has non-empty interior and then to the case $0 \in C^0$. Let $\|x_n\| \to \infty, \{x_n\} \subseteq C$. Let $\frac{1}{\|x_{n_j}\|}x_{n_j} \to x$. Let $t > 0$. We complete the proof by showing that $tx \in C$. We have $tx = \frac{1}{2}(y_j + (2tx-y_j))$ where $y_j = \frac{2t}{\|x_{n_j}\|}x_{n_j}$. Note that $y_j \to 2tx$ as $j \to \infty$ so $y_j - 2tx \in C$ whenever is j sufficiently large. Also $y_j = \frac{2t}{\|x_{n_j}\|}x_{n_j} + (1-\frac{2t}{\|x_{n_j}\|})0 \in C$ if $\frac{2t}{\|x_{n_j}\|} < 1$ which is true whenever is j sufficiently large.

Problem 618

Find all complex Borel measures μ on $[0,1]$ such that $f \to \int f d\mu$ is a homomorphism on the alegbra $C[0,1]$

(i.e. $\int f d\mu \int g d\mu = \int fg d\mu \ \forall f,g \in C[0,1]$). Find all complex Borel measures μ on $[0,1]$ such that $\int f d\mu \neq 0$ whenever $f:[0,1] \to \mathbb{C}\backslash\{0\}$ is continuous and $\mu([0,1]) = 1$.

If $C \subseteq C[0,1]$ is closed then there exist continuous functions $f_n : [0,1] \to [0,1]$ such that $f_n = 1$ on C and $f_n(x) = 0$ if $d(x,C) \geq \frac{1}{n}$. Since $f_n \to I_C$ pointwise and $\int f_n d\mu \int f_n d\mu = \int f_n^2 d\mu \ \forall n$ we get $\mu^2(C) = \mu(C)$ for every closed set C. Regularity of μ implies that the same equation holds for all Borel sets C.In particular $\mu(E) = 0$ or 1 for every Borel set E. Assume that μ is not the zero measure. For each n there is a unique j_n such that $\mu([\frac{j_n-1}{n}, \frac{j_n}{n}]) = 1$. Now $\bigcap_n [\frac{j_n-1}{n}, \frac{j_n}{n}]$ is a singleton $\{c\}$ and $\mu\{c\} = 1$ which implies $\mu = \delta_c$. Conversely, $\int f d\mu \int g d\mu = \int fg d\mu \ \forall f, g \in C[0,1]$ if $\mu = \delta_c$ for some c. By the theorem of Gleason, Kahane and Zelazko (see Theorem 10.9 of Functional Analysis by Rudin) the answer to the second part is the same.

Problem 619

If X is an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subseteq \mathcal{F}$ is a sigma algebra such that X and $E(X|\mathcal{G})$ have the same distribution show that $E(X|\mathcal{G}) = X$ almost surely.

Remark: the conclusion says that X is measurable w.r.t. $\mathcal{G}$ (rather its completion). Thus $E(X|\mathcal{G})$ and X have different distribution unless there is no real conditioning involved!

Let $\phi(x) = \begin{cases} x - 1 + e^{-x} \text{ if } x \geq 0 \\ -x - 1 + e^{x} \text{ if } x < 0 \end{cases}$. Then $\phi'(x)$ is strictly increasing on $\mathbb{R}$. Hence $\phi(y) \geq \phi(x) + \phi'(x)(y-x) \ \forall x, y$ and strict inequality holds if $x \neq y$. [If, for example, $y > x$ then $\frac{\phi(y)-\phi(x)}{y-x} > \phi'(x)$ because $\frac{\phi(y)-\phi(x)}{y-x} = \phi'(t)$ for some $t \in (x, y)$ and $\phi'(t) > \phi'(x)$]. Note that $|\phi(x)| \leq 2 + |x| \ \forall x$ so $\phi(X)$ is integrable and also that ϕ' is bounded. Also, $\phi(X) \geq \phi(E(X|\mathcal{G})) + \phi'(E(X|\mathcal{G}))\{X - E(X|\mathcal{G})\}$ with strict inequality except when $X = E(X|\mathcal{G})$. However the two sides of the inequality have the same mean because $\phi(E(X|\mathcal{G}))$ has the same distribution as $\phi(X)$. It follows that equality holds almost surely and hence $X = E(X|\mathcal{G})$ almost surely.

Problem 620

If $E(X_n|\mathcal{G}) \to 0$ a.s. and each X_n is a non-negative random varable show that $X_n \to 0$ in probability.

Let $Y_n = \min\{X_n, 1\}$. Then $E(Y_n|\mathcal{G}) \leq E(X_n|\mathcal{G}) \to 0$ abd $0 \leq E(Y_n|\mathcal{G}) \leq 1$ so $E(E(Y_n|\mathcal{G})) \to 0$ or $EY_n \to 0$. Hence $Y_n \to 0$ in probabiliy. For $0 < \varepsilon < 1$, $X_n > \varepsilon$ implies $Y_n > \varepsilon$ so $\{X_n\} \to 0$ in probability.

Problem 621

Suppose $E(X|Y) = EX$. Does it follow that X and Y are independent?

No. If $Y = I_A - I_B$ and $\int_A X dP = \int_B X dP = \int X dP = 0$ then $E(X|Y) = EX$. However X and Y need not be independent. [On $[0,1]$ with Lebesgue mesure let $X = I_E - I_F + I_G - I_H$ where $E = (0, \frac{1}{8}), F = I_{(\frac{1}{8}, \frac{1}{4})}, G = I_{(\frac{6}{8}, \frac{7}{8})}, H = I_{(\frac{7}{8}, 1)}$ and $Y = I_{(0, \frac{1}{4})} - I_{(\frac{3}{4}, 1)}$. Note that $P\{X = 1, Y = 1\} \neq P\{X = 1\}P\{Y = 1\}$].

Problem 622

There exist sequences $\{a_n\}, \{b_n\} \subseteq (0, \infty)$ such that $a_{n+1} < a_n, b_{n+1} < b$ $\forall n, \sum_{n=1}^{\infty} a_n = \infty, \sum_{n=1}^{\infty} b_n = \infty$ but $\sum_{n=1}^{\infty} \min\{a_n, b_n\} < \infty$.

We shall construct positive integers $k_N (N \geq 1)$ and positive numbers $a_n' s$ and $b_n' s$ with the following properties: a_j and b_j are independent of N, $a_1 > a_2 > ... > a_{k_N}$, $b_1 > b_2 > ... > b_{k_N}, \sum_{j=1}^{k_N} a_j \geq N, \sum_{j=1}^{k_N} b_j \geq N, a_j \leq \frac{1}{j}(1 \leq j \leq k_N), b_j \leq \frac{1}{j}(1 \leq j \leq k_N)$ and $\min\{a_j, b_j\} \leq \frac{1}{j^2}(1 \leq j \leq k_N)$.

Of course $\{a_n\}$ and $\{b_n\}$ would then satisfy our requirements. We start with $a_1 = b_1 = k_1 = 1$. Suppose we have constructed $k_1, ..., k_N, a_j(j \leq k_N)$ and $b_j(j \leq k_N)$. We use the following steps to construct k_{N+1} and $a_j, b_j(k_N < j \leq k_{N+1})$.

Step 1: pick $l > \max\{(\frac{1}{a_{k_N}} - 1), k_N, (\frac{1}{\sqrt{b_{k_N}}} - 1)\}$.

Step 2: pick a positive integer r such that $\sum_{\rho=1}^{r} \frac{1}{\rho+l} \geq N + 1$.

Step 3: pick a positive integer m such that $m > \max\{\sqrt{l+r} - 1), \{(l+r)^2 - 1\}, k_N + r\}$.

Step 4: pick a positive integer s such that $\sum_{\rho=1}^{s} \frac{1}{\rho+m} \geq N + 1$.

Let $k_{N+1} = k_N + r + s$.

Let $a_{k_N+j} = \frac{1}{l+j}$ if $1 \leq j \leq r, a_{k_N+j+p} = (\frac{1}{m+p})^2$ if $1 \leq p \leq s, b_{k_N+j} = (\frac{1}{l+j})^2$ if $1 \leq j \leq r, b_{k_N+j+p} = \frac{1}{m+p}$, if $1 \leq p \leq s$.

Remark: if we drop the requirement that $\{a_n\}$ and $\{b_n\}$ are decreasing sequences the solution becomes trivial: let $a_n = \frac{1}{n^2}$ or $\frac{1}{n}$ according as n is even or odd and $b_n = \frac{1}{n}$ or $\frac{1}{n^2}$ according as n is even or odd.

Problem 623

Let $t, t_1, t_2, ... \in \mathbb{R}$. Show that $t_n \to t$ if and only if $\hat{f}(t_n) \to \hat{f}(t)$ $\forall f \in L^1(\mathbb{R})$.

'only if' part is by continuity of the Fourier transform. For the 'if' part let $f(x) = e^{iax}e^{-x^2/2}$ (where a is a real number) to conclude that $e^{iat_n}e^{-t_n^2/2} \to e^{iat}e^{-t^2/2}$. This implies $e^{-t_n^2/2} \to e^{-t^2/2}$ and $e^{iat_n} \to e^{iat}$. Also $e^{iat_n}e^{-t_n^2/2} \to e^{iat}e^{-t^2/2}$ implies that $\{t_n\}$ is bounded. It follows from $e^{iat_n} \to e^{iat}$ that t is the only limit point of $\{t_n\}$.

Problem 624

If μ is a complex Borel measure on a locally compact Hausdorff space X such that $\|\mu\| = \mu(X)$ show that μ is a positive measure.

Let $\nu = |\mu|$ and $\phi = \frac{d\mu}{d\nu}$. Then $|\phi| = 1$ a.e. $[\nu]$ and $\int \phi d\nu = \mu(X) = \|\mu\| = \nu(X)$. Hence $\nu(X) = \operatorname{Re} \int \phi d\nu = \int \operatorname{Re} \phi d\nu \leq \int 1 d\nu = \nu(X)$. Hence $\operatorname{Re} \phi = 1$ a.e. $[\nu]$. Since $|\phi| = 1$ a.e. $[\nu]$ this implies $\phi = 1$ a.e. $[\nu]$ so $\mu = \nu$.

Problem 625

Prove that $\{\hat{f} : f \in L^1(\mathbb{R})\} = \{g * h : g, h \in L^2(\mathbb{R})\}$.

Remark: if $g, h \in L^2(\mathbb{R})$ then $g * h$ is well-defined by Holder's inequality. Above statement implies that it is continuous and vanishes at $\pm\infty$.

Let $f \in L^1(\mathbb{R})$. There exist functions f_1, f_2 in $L^2(\mathbb{R})$ such that $f = f_1 f_2$. There exist functions ϕ_1, ϕ_2 in $L^2(\mathbb{R})$ such that $f_1 = \hat{\phi}_1, f_2 = \hat{\phi}_2$. Let $g(x) = \phi_1(-x)$ and $h(x) = \phi_2(-x)$. Then $(g * h)\hat{}\,(t) = \hat{g}(t)\hat{h}(t) = \hat{\phi}_1(-t)\hat{\phi}_2(-t) = f_1(-t)f_2(-t) = f(-t)$. Taking Fourier transforms we get $(g*h)(-x) = \hat{f}(-x)$ so $\hat{f} = g * h$. Conversely, if $g, h \in L^2(\mathbb{R})$ then $g = \hat{g}_1, h = \hat{h}_1$ for some $g_1, h_1 L^2(\mathbb{R})$.

Let $f = g_1 h_1$. Then $f \in L^1(\mathbb{R})$ and $(g * h)\hat{}\,(x) = \hat{g}(x)\hat{h}(x) = f(-x) = f\hat{}\,\hat{}\,(x)$ and hence $g * h = \hat{f}$.

Problem 626

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ be a sub-sigma field of $\mathcal{F}$, and let X and Y be two random vriables on $(\Omega, \mathcal{F}, P)$. Suppose suppose there is a set $E \in \mathcal{F}$ such that $P(E) = 1$ and $P(X^{-1}(A)|\mathcal{G})(\omega) = I_A(Y(\omega))$ for every Borel set A whenever $\omega \in E$. Prove that $X = Y$ a.s. and that X is measurable w.r.t the $P-$ completion of $\mathcal{G}$.

We prove that $P\{X^{-1}(A) \cap Y^{-1}(A^c)\} = 0$ for any Borel set A. This implies that $X = Y$ a.s. and hence that $P(X^{-1}(A)|\mathcal{G}) = I_A(X)$ a.s. which implies $X^{-1}(A) \in \mathcal{G}$ for each A if $\mathcal{G}$ is complete w.r.t. P. The hypothesis implies that

$Y^{-1}(A^c) = (Y^{-1}(A))^c \in \mathcal{G}$ and hence $\int\limits_{Y^{-1}(A^c)} P(X^{-1}(A)|\mathcal{G})dP = P\{X^{-1}(A) \cap Y^{-1}(A^c)\}$. But the left side of this equation is $\int\limits_{Y^{-1}(A^c)} I_A(Y)dP = P\{Y^{-1}(A) \cap Y^{-1}(A^c)\} = 0$. [We used the fact that if ν is a probability measure on $\mathbb{R}^2$ such that $\nu(A \times A^c) = 0 \; \forall A$ Borel in $\mathbb{R}$ then $\nu(\Delta) = 1$, where Δ is the diagonal: $\Delta = \{(x.x) : x \in \mathbb{R}^2\}$. Indeed Δ^c is the union of the sets $(-\infty, r) \times (-\infty, r)^c$ and $(r, \infty)) \times (r, \infty)^c$ as r varies over $\mathbb{Q}$].

Problem 627

Let $X_n \to 0$ a.s. and assume that $p\{X_n = 0\} = 0$ for each n. Show that there exists a measurable function $f : \mathbb{R} \to (0, \infty)$ such that $\sum f(X_n) < \infty$ a.s.

Let Y_j be the number of positive integers n for which $\frac{1}{j} \leq |X_n| < \frac{1}{j-1}$. [Define $Y_j(\omega)$ to be 0 if $X_n(\omega) \nrightarrow 0$]. Then Y_j is a positive random variable for each j. Claim: there exists $a_j > 0$ $(j = 1, 2, \ldots)$ such that $\sum\limits_j a_j Y_j < \infty$ a.s.. To see this just choose a_j such that $P\{a_j Y_j > \frac{1}{j^2}\} < \frac{1}{j^2}$ and note that $\sum\limits_j P\{a_j Y_j > \frac{1}{j^2}\} < \infty$ which implies $a_j Y_j \leq \frac{1}{j^2}$ eventually, with probability 1. The claim is proved. Now let $f(x) = a_j$ on $\{x : \frac{1}{j} \leq |x| < \frac{1}{j+1}\}$, $j = 2, 3, \ldots$. [we can take f to be 1 on $\{x : \frac{1}{2} \leq |x| < \infty\} \cup \{0\}$]. Then $\sum f(X_n) = \sum\limits_j \sum\limits_{\frac{1}{j} \leq |X_n| < \frac{1}{j-1}} a_j = \sum\limits_j a_j Y_j < \infty$ a.s..

Problem 628

Call a subset A of $\mathbb{R}$ nicely covered (nc) if there is a sequence of open sets $\{U_n\}$ such that $A \subseteq U_n$ for each n and any open set V that contains A necessarily contains some U_n. Show that A is nc if and only if it is the union of a compact set and an open set

If $A = K \cup U$ where K is compact and U is open let $U_n = \{x : d(x, K) < \frac{1}{n}\} \cup U$. Then U_n is open, contains A and if V is an open set with $A \subseteq V$ then $K \subseteq V$ and $U \subseteq V$. It follows that $\{x : d(x, K) < \frac{1}{n}\} \subseteq V$ for some n and $U_n \subseteq V$ for that n. Hence A is nc. Coversely suppose A is nc. If we show that $A \backslash A^0$ is compact the proof would be complete because $A = A^0 \cup (A \backslash A^0)$. Suppose $A \backslash A^0$ is not compact. Let $\{x_n\}$ be a sequence in $A \backslash A^0$ which has no limit point in $A \backslash A^0$. If $\{x_n\}$ has a limit point, say x, in A then $x \notin A \backslash A^0$ so $x \in A^0$. But then $x_n \in A^0$ for n sufficiently large which is a contradiction. Hence $\{x_n\}$ has no limit point in A. By hypothesis there is a sequence of open sets $\{U_n\}$ such that $A \subseteq U_n$ for each n and any open set V that contains A necessarily contains some U_n. There exists $y_n \in U_n \backslash A$ such that $d(x_n, y_n) < \frac{1}{n}$. [This is becase $x_n \in \partial A \cap U_n$]. Let $V = \bar{B}^c$ where $B = \{y_1, y_2, \ldots\}$. Note that V is open and $A \subseteq V$. This is because any limit point of $\{y_n\}$ is also a limit point of $\{x_n\}$ and hence it does not belong to A. proving that $\bar{B} \subseteq A^c$ or $A \subseteq V$]. It follows that $U_n \subseteq V$ for some n. However $y_n \in U_n$ but $y_n \notin V$. This completes the proof.